SHORT

Continuous Functions Without Derivatives

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

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ENTITLED Continuous Functions without Derivatives.

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## Introduction.

In Chapter I. I shall define a function of one real variable, and discuss continuity and discontinuity, giving examples to illustrate a continuous function and the different kinds of discontinuous functions.

In Chapter II. I shall define a derivative, give a brief historical sketch of its developement, and discuss the conditions necessary for the existence of a derivative.

In Chapter III, I shall give examples of functions, which are continuous but do not have a derivative at a single point or a finite number of points. Then show, through the so called condensation of singularities, how this property can be made to exist at an infinite number of points, and finally present examples which have not a derivative at any point in an interval.

In Chapter IV, I shall take up

the geometrical representation of continuous functions, which at no point possess a derivative and treat them according to methods used by Steinitz and Peano.

# Chapter I.
## Continuity and Discontinuity.

§1. Definition of a function of one real variable.— $y$ is a function of $x$ within a given interval, if for every value for which $x$ is defined there exists a definite value of $y$ depending upon $x$. Thus $y$ cannot be considered as a function of $x$ if we know only,

$$y = \sin \frac{1}{x}$$

in an interval which contains the point $x = 0$; for it has no definite value at this point. It would, however, be a function of $x$, if we had said,

$$y = \sin \tfrac{1}{x}, \text{ for } x \neq 0$$
$$y = 0 \quad , \text{ for } x = 0.$$

For $y$ would then possess a definite value for every value of $x$ within the interval.

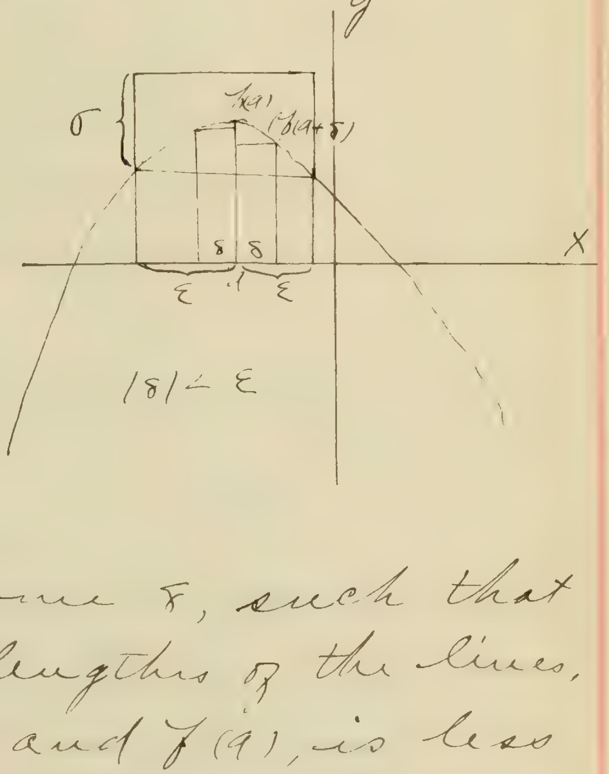
§2. Condition for Continuity at a Point.— If $f(x)$ has a definite value for every value of $x$ in the interval,

which we shall call $(\alpha, \beta)$, we say it is a continuous function of $x$ at the point $a$, or for $x = a$, at which its value is $f(a)$, if for every arbitrarily small positive number $\sigma$ different from zero, there exists another positive number $\varepsilon$ such that for every value of $\delta$ numerically smaller than $\varepsilon$, we have

$$|f(a \pm \delta) - f(a)| < \sigma.$$

This can be shown geometrically as follows:

Let the curve represent $f(x)$. If we choose any arbitrarily small value $\sigma$ and lay it off from $f(a)$ on the perpendicular to the $x$ axis, then we must be able to choose some $\delta$, such that the differences in the lengths of the lines, representing $f(a \pm \delta)$ and $f(a)$, is less than the distance $\sigma$.




In other words $f(x)$ is a continuous function for $x = a$, if

$$\lim_{h=0} f(a \pm h) = f(a)$$

where $h$ is an arbitrarily small positive number. Or $f(x)$ is a continuous function for $x = a$, if

$$f(a+h) - f(a)$$
$$\text{and } f(a-h) - f(a)$$

can be made as small as we choose. Or $f(x)$ is a continuous function for $x = a$, if the limits of the value of $f(x)$ right and left of the point $a$ are equal to $f(a)$.

Example: Given the function

$$y = x \sin\tfrac{1}{x}, \quad \text{for } x \neq 0$$
$$y = 0 \quad\quad\quad , \text{for } x = 0.$$

This is a continuous function at the point zero; for

$$\lim_{\delta=0} (0+\delta)\sin\left(\frac{1}{0+\delta}\right) - 0.\sin\left(\frac{1}{0}\right) = \lim_{\delta=0} \delta \sin\frac{1}{\delta}$$
$$= 0.$$

<u>§ 3. Continuity of a function in an interval.</u>— If the function $f(x)$ is continuous at every point in the interval $(\alpha, \beta)$, it is said to be continuous in the interval.

Example: Given the function
$$y = \sin x.$$
This function is continuous in the interval
$$0 \leqq x_0 \leqq \frac{\pi}{2}.$$
For
$$\underset{\delta \doteq 0}{L} \sin(x_0 \pm \delta) - \sin x_0 = \underset{\delta \doteq 0}{L} (\sin x_0 \cos\delta \pm \cos x_0 \sin\delta) - \sin x_0$$
$$= 0.$$

§ 4. Discontinuity of a function at a point. — The function $f(x)$ is discontinuous at the point $a$, if for an arbitrarily small positive value of $\sigma$ there exists no positive value of $\varepsilon$ such that for $\delta$, numerically smaller than $\varepsilon$
$$|f(a \pm \delta) - f(a)| < \sigma.$$
Or $f(x)$ is discontinuous if the value $f(a+h)$ right of $a$ and the value $f(a-h)$ left of $a$ have no definite limits as $h \doteq 0$. Or if they have definite limits and these limits are different on the two sides of $a$. Or if the limits right and left are definite and equal but different from $f(a)$.

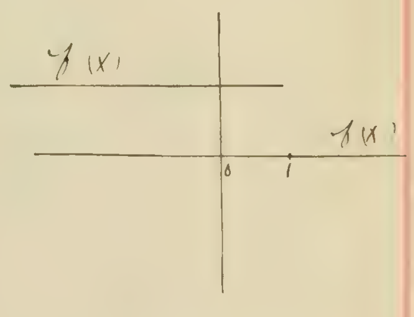
If $a$ is one of the end points of the interval, we can speak only of the continuity or discontinuity on one side of the end point, and therefore consider only $f(a+h)$ or $f(a-h)$.

§5. Discontinuity of a function to the right or left of a point.- If the point $a$ is a point at which $f(x)$ is discontinuous and the value $f(x)$ on one side of $a$ has $f(a)$ for its limit, but on the other side the value $f(x)$ has no definite limit or has a definite limit different from $f(a)$ we say that $f(x)$ is continuous on one side of $a$, to the right or left and discontinuous on the other, viz. to the left or right.

Example: Given the function

$$f(x) = 1, \text{ for } x < 1$$
$$f(x) = 0, \text{ for } x \geqq 1.$$

Then $f(x)$ is continuous to the right of the point 1 and discontinuous to the left.



§ 6. Discontinuity of the first kind.— If the point $a$ is not an end point of the interval $(\alpha, \beta)$ and the limits of $f(a+h)$ and $f(a-h)$ are definite and equal to $A$, but different from $f(a)$ then the continuity of the function can be restored by taking the value of $f(a)$ as $A$.

Example:— Given the function

$$f(x) = \sum_{n=0}^{\infty} \sin^2 x \cos^{2n} x,$$

for all values of $x$ in the interval $0 \leqq x \leqq \pm 2\pi$. The function is discontinuous at the points $0, \pm\pi, \pm 2\pi$; for

$$f(x) = 1$$

for all values of $x$, except $x = 0, \pm\pi, \pm 2\pi$, for which

$$f(x) - 0.$$

We can however, restore the continuity of the function at these points by defining our function as follows:—

$$f(x) = \sum_{n=0}^{\infty} \sin^2 x \cos^{2n} x$$

for all values of $x$ in the interval $(0, \pm 2\pi)$ except for $x = 0, \pm\pi, \pm 2\pi$, and for these values

$$f(x) = 1.$$

If a discontinuity of $f(x)$ takes place on one side of the point and this discontinuity is such that the value of $f(x)$ has a definite limit on this side of the point, the discontinuity is said to be of the first kind. (Dini p. 51.)

Discontinuities which can be removed by changing the value of the function at the point are always of the first kind on both sides of the point.

§7 Discontinuity of the second kind.— If a discontinuity occurs on one side of the point a and this discontinuity is such that the value of $f(x)$ has no definite limit on this side of the point, the discontinuity is said to be of the second kind. For example, if on one side of the point a, $f(x)$ makes an infinite number of oscillations of a given amplitude we have a discontinuity of the second kind right or left of the point, since $f(x)$ has no definite limit.

Example: Given the function

$$f(x) = \sin\frac{1}{x-a} \quad \text{for } x \neq a$$

$$f(x) = 0, \text{ for } x = a.$$

This function has a discontinuity of the second kind right and left of the point $a$ and oscillates between $+1$ and $-1$ as $x \doteq a$ from either side.

If the function $f(x)$ is discontinuous at the point $a$ which is not an end point of the given interval, it can be continuous on one side and ~~have~~ on the other side a discontinuity of either the first or second kind; or if it is discontinuous on both sides of the point, it can have on both sides a discontinuity of either the first or second kind, or on ~~one~~ a discontinuity of the first kind and on the other, one of the second kind.

Example: Given the function

$$f(x) = x^2 + \sin\left|\frac{1}{x}\right| + \sin\frac{1}{x}.$$

This function has a discontinuity, on the right of $x=0$, of the second kind and is continuous on the left of $x=0$.

(See Thesis of F. A. Smith '01)

Example: Given the function

$$f(x) = \frac{1}{1+e^{\frac{1}{x}}} + \sin\left|\frac{1}{x}\right| + \sin\frac{1}{x}.$$

This function has a discontinuity of the first kind on the left of $x=0$, and a discontinuity of the second kind on the right.

By changing the value of the function at the point in question the discontinuity can always be removed at least on one side if the discontinuity is of the first kind, but never if it is of the second kind. If the discontinuity occurs at an end point of the interval and is of the first kind it can always be removed by changing the value of the function at that point.

§ 8. Discontinuity in an interval.— Functions may have discontinuities not only at a single point but also at a finite number of points, at an infinite number of points, and even at every point of the interval.

Example: Given the function

$$f(x) = \frac{1}{1+e^{1/(\sin 2n\pi x)}}.$$

This function is discontinuous at as

large a number of rational points as we please, only that the number does not become infinite, and is continuous at every other point.

Example: Given the function

$$f(x) = \underset{n \doteq \infty}{L} \sum_{k=1}^{k=\infty} \left( \frac{\sin^2 \lfloor\!\underline{k}\, \pi x}{\lfloor\!\underline{k}} \right)^{\frac{1}{n}}.$$

This function is discontinuous at every rational point and continuous at every irrational point.

Example: Given the function

$$f(x) = \underset{n \doteq 0}{L} \left[ \underset{t \doteq 0}{L} \left( \frac{\sin^2 \lfloor\!\underline{n}\, \pi x}{\sin^2 \lfloor\!\underline{n}\, \pi x + t^2} \right) \right].$$

For rational values of $x$

$$f(x) = 0,$$

and for irrational values of $x$,

$$f(x) = 1.$$

The function is therefore discontinuous at every point.

# Chapter II

## Derivatives and the Conditions for their Existence.

§1. Definitions.— If $f(x)$ is a function which is finite and continuous at every point of the interval $(\alpha, \beta)$, and if $x_0$ is a point in this interval and if $\underset{\delta \doteq 0}{L} \dfrac{f(x_0+\delta) - f(x_0)}{\delta}$ is finite and has a definite sign, independent of the sign of $\delta$, then $f(x)$ is said to have a definite, finite derivative at the point $x_0$. If $\underset{\delta \doteq 0}{L} \dfrac{f(x_0+\delta) - f(x_0)}{\delta}$ is infinite and has a definite sign, independent of the sign of $\delta$, $f(x)$ is said to have a definite infinite derivative at the point $x_0$. If however $\underset{\delta \doteq 0}{L} \dfrac{f(x_0+\delta) - f(x_0)}{\delta}$ is not determined and independent of the sign of $\delta$ then the derivative of $f(x)$ at the point $x_0$ is said to be wholly indeterminate or not to exist. This includes

several cases. (1) Where the limit is finite but has no definite sign, that is where $\underset{\delta \doteq 0}{L} \frac{f(x_0+\delta)-f(x_0)}{\delta}$ oscillates between finite limits in the neighborhood of $x_0$. (2) Where $\underset{\delta \doteq 0}{L} \frac{f(x_0+\delta)-f(x_0)}{\delta}$ oscillates between infinite limits. (3) Where $\underset{\delta \doteq 0}{L} \frac{f(x_0+\delta)-f(x_0)}{\delta}$ is finite and definite but different for positive and negative $\delta$. (4) Where $\underset{\delta \doteq 0}{L} \frac{f(x_0+\delta)-f(x_0)}{\delta}$ is $+\infty$ for positive or negative $\delta$ and $-\infty$ for negative or positive $\delta$. (5) Where $\underset{\delta \doteq 0}{L} \frac{f(x_0+\delta)-f(x_0)}{\delta}$ is finite for positive or negative $\delta$ and infinite for negative or positive $\delta$.

The limit of $\frac{f(x_0+\delta)-f(x_0)}{\delta}$, as $\delta$ approaches zero through positive values, is called the derivative of $f(x)$ right of the point $x_0$, and the limit of $\frac{f(x_0+\delta)-f(x_0)}{\delta}$, as $\delta$ approaches zero through negative

values, is called the derivative left of the point $x_0$. If the point $x_0$ is an end point of the interval, we can consider only the derivative right or the derivative left of it.

§2. Historical. Who was the discoverer of derivatives has been the subject of a great deal of discussion. In a manuscript of Leibniz dated 1673 he sets forth the principle of the derivative in dealing with tangents and the quadrature of curves. In a paper dated 1675, Leibniz uses the symbol $dx$ but he does not use the term differential, but difference. In a paper of July 11, 1677, Leibniz gave correct rules for the differentiation of sums, products, quotients, powers, and roots. In 1684 Leibniz published, in the Leipzig acts, his first paper on Calculus. Newton had begun using his notation of fluxions in 1666, and the English mathematicians claimed that Leibniz on his visit to London in 1673 had gotten his

ideas from Newton's notation of fluxions and after changing the symbols and definitions of terms had published them as his own. From Leibnitz' manuscript, however, it appears that he independently invented the derivative and while Newton must be given credit for its earliest discovery Leibnitz was the first to give its full benefits to the world.

Up to the year 1806, it had been a generally accepted idea that a function continuous in an entire interval had a derivative at every point of that interval. This idea was formed from the geometrical representation of the derivative as a tangent. In this year Ampere raised the question why should a continuous function have a derivative at every point of the interval for which it is defined, and attempted to give an analytical proof of it, but succeeded only in demonstrating that the derivative of a continuous function, which was not constant,

could not always be $+\infty$ or $-\infty$. Riemann's conception of an integral first led to functions which were continuous but had no derivative at an infinite number of points and at all points. Weierstrass was the first to give a well completed, clear proof of this and since this problem numerous others have been worked out with similar results.

§3. Conditions for a derivative. That continuity is a necessary condition for a definite derivative is plainly evident; for $\lim_{\delta=0} \frac{f(x_0+\delta)-f(x_0)}{\delta}$ would

always become infinite if we did not have the condition

$$|f(x_0+\delta)-f(x_0)|<\sigma,$$

where $\delta$ and $\sigma$ are defined as in Chapter I. The above condition is the condition for continuity, and therefore no function can have a derivative at a given point, if it is not continuous at that point. That continuity is not a sufficient condition for

a derivative is shown by the fact that functions are known which are continuous in an interval and yet have no derivative;—
(1) at a single point in the interval.
(2) at a finite number of points in the interval.
(3) at an infinite number of points in the interval.
(4) at every point in the interval.

That other conditions besides continuity are necessary for the existence of a derivative is evident, but what these conditions are or whether there is a necessary and sufficient condition has as yet never been demonstrated.

# Chapter III
## Functions which are Continuous but have no Derivatives.

### §1. Continuous functions having no derivatives at a single point.

Example: Given the function

$$f(x) = x \sin\tfrac{1}{x} \quad , \text{ for } \neq 0,$$
$$f(x) = 0 \qquad , \text{ for } x = 0.$$

The function $f(x)$ is continuous at the point 0; for

$$\underset{\delta \doteq 0}{L} \ (0 \pm \delta) \sin\left(\frac{1}{0 \pm \delta}\right) - 0 . \sin\frac{1}{0} = 0.$$

However $f(x)$ has no derivative at the point 0; for

$$\underset{\delta \doteq 0}{L} \ \frac{(0 \pm \delta) \sin\left(\frac{1}{0 \pm \delta}\right) - 0 . \sin\frac{1}{0}}{\pm \delta} = \sin\frac{1}{0}$$
$$= \sin \infty$$

and hence oscillates between +1 and -1, never having a definite value.

Example 2: Given the function

$$f(x) = x \cos\tfrac{1}{x} \ , \text{ for } x \neq 0,$$
$$f(x) = 0 \qquad , \text{ for } x \doteq 0.$$

The function $f(x)$ is continuous at the point 0; for

$$\underset{\delta \doteq 0}{L}\ (0 \pm \delta)\cos\left(\frac{1}{0\pm\delta}\right) - 0.\cos\frac{1}{0} = 0.$$

However $f(x)$ has no derivative at the point 0; for

$$\underset{\delta \doteq 0}{L}\ \frac{(0\pm\delta)\cos\left(\frac{1}{0\pm\delta}\right) - 0.\cos\frac{1}{0}}{\delta} = \cos\frac{1}{0}$$
$$= \cos\infty,$$

and therefore oscillates between +1 and -1 and never has a definite value.

Example 3! Given the function
$$f(x) = x^{2/3},$$
for all values of $x$. $f(x)$ is continuous at the point 0; for
$$\underset{\delta \doteq 0}{L}\ (0\pm\delta)^{2/3} - (0)^{2/3} = 0.$$
However $f(x)$ has no derivative at the point 0; for
$$\underset{\delta \doteq 0}{L}\ \frac{(0+\delta)^{2/3} - 0^{2/3}}{\delta} = \underset{+\delta \doteq 0}{L}\ \frac{1}{\delta^{1/3}}$$
$$= +\infty,$$
$$\underset{\delta \doteq 0}{L}\ \frac{(0-\delta)^{2/3} - 0^{2/3}}{-\delta} = \underset{\delta \doteq 0}{L}\ -\frac{1}{\delta^{1/3}}$$
$$= -\infty.$$
The values are not independent of the sign of $\delta$ and we can have no derivative at the point 0.

Example 4! Given the function
$$f(x) = |x|, \text{ for } x \gtrless 0,$$

$$f(x) = -x, \text{ for } x < 0.$$

Then $f(x)$ is continuous for $x = 0$; for

$$\underset{\delta \doteq 0}{L} (0+\delta) - (0) = 0,$$

$$\underset{\delta \doteq 0}{L} -(0-\delta) - (-0) = 0.$$

However $f(x)$ has no derivative at the point 0; for

$$\underset{\delta \doteq 0}{L} \frac{(0+\delta) - 0}{\delta} = 1$$

$$\underset{\delta \doteq 0}{L} \frac{-(0-\delta) - (-0)}{-\delta} = -1.$$

The values are not independent of the sign of $\delta$, and therefore the derivative at the point 0 does not exist.

Example 5: Given the function

$$f(x) = x \sin(\log x^2)$$

for all values of $x$. Then $f(x)$ is continuous at the point 0; for

$$\underset{\delta \doteq 0}{L} (0 \pm \delta) \sin(\log(0 \pm \delta)^2) - 0 \sin(\log 0^2) = \underset{\delta \doteq 0}{L} \delta \sin(\log \delta^2)$$
$$= 0.$$

However $f(x)$ has no derivative at the point 0; for

$$\underset{\delta \doteq 0}{L} \frac{(0 \pm \delta) \sin(\log(0 \pm \delta)^2) - 0 \sin(\log 0^2)}{\pm \delta} = \underset{\delta \doteq 0}{L} \frac{\pm \delta \sin(\log \delta^2)}{\pm \delta}$$

$$= \sin(-\infty),$$

which may have any value between +1 and -1 and therefore the derivative at the point 0 does not exist.

Example 6: Given the function

$$f(x) = \frac{1}{\log x^2},$$

for all values of $x$. Then $f(x)$ is continuous at the point 0; for

$$\underset{\delta \doteq 0}{L} \frac{1}{\log(0 \pm \delta)^2} - \frac{1}{\log 0^2} = \underset{\delta \doteq 0}{L} \frac{1}{\log \delta^2}$$
$$= 0.$$

However $f(x)$ has no derivative at the point 0; for

$$\underset{\delta \doteq 0}{L} \frac{\frac{1}{\log(0+\delta)^2} - \frac{1}{\log 0^2}}{\delta} = \underset{\delta \doteq 0}{L} \frac{1}{\delta \log \delta^2}$$
$$= +\infty^*,$$

$$\underset{\delta \doteq 0}{L} \frac{\frac{1}{\log(0-\delta)^2} - \frac{1}{\log 0^2}}{-\delta} = \underset{\delta \doteq 0}{L} - \frac{1}{\delta \log \delta^2}$$
$$= -\infty.$$

These values are not independent of the sign of $\delta$, and therefore the derivative at the point 0 does not exist.

## §2. Continuous functions having no derivative at a finite number of points.

* see Chrystal vol II. p 86.

Example 1: Given the function $f(x) = \sin x \sin\left(\frac{1}{\sin x}\right)$, for $0 \leqq x \leqq 2\pi$. $f(x)$ is a continuous function at the points $0, \pi, 2\pi$: for, when $k = 0, 1, 2$,

$$\underset{\delta \doteq 0}{L} \sin(\pm\delta + k\pi)\sin\left(\frac{1}{\sin(\pm\delta + k\pi)}\right) - \sin k\pi \sin\left(\frac{1}{\sin k\pi}\right)$$

$$= \underset{\delta \doteq 0}{L} (-1)^k \sin(\pm\delta)\sin\left(\frac{(-1)^k}{\sin(\pm\delta)}\right) - 0.\sin\frac{1}{0} = 0.$$

However $f(x)$ has no derivative at the points $0, \pi, 2\pi$; for

$$\underset{\delta \doteq 0}{L} \frac{\sin(\pm\delta + k\pi)\sin\left(\frac{1}{\sin(\pm\delta + k\pi)}\right) - \sin k\pi \sin\left(\frac{1}{\sin k\pi}\right)}{\delta}$$

$$= \underset{\delta \doteq 0}{L} \frac{(-1)^k \sin\pm\delta \sin\left(\frac{(-1)^k}{\sin\pm\delta}\right)}{\delta}$$

$$= \underset{\delta \doteq 0}{L} (-1)^k \frac{\sin\delta}{\delta} \cdot \underset{\delta \doteq 0}{L} \sin\left(\frac{(-1)^k}{\sin\delta}\right).$$

But $\underset{\delta \doteq 0}{L} \frac{\sin\pm\delta}{\pm\delta} = 1$

and $\underset{\delta \doteq 0}{L} \sin\left(\frac{(-1)^k}{\sin\pm\delta}\right) = \sin(\pm\infty)$.

$$\therefore \underset{\delta \doteq 0}{L} \frac{f(x\pm\delta) - f(x)}{\pm\delta} = \sin(\pm\infty).$$

$\sin(\pm\infty)$ has no definite value but may have any value between $+1$ and $-1$, and consequently the function has no derivative at the points $0, \pi, 2\pi$.

Example 2: Given the function

$$f(x) = \sin \lfloor n \pi x \sin\left(\frac{1}{\sin \lfloor n \pi x}\right),$$

where $n$ is a finite integer less than some definite value $y$. $f(x)$ is then a continuous function but has no derivative at any point $x = \frac{p}{q}$, where $p$ and $q$ are finite integers; that is, it has no derivative at as large a number of rational points as we choose so long as that number does not become infinite.

Let $k = p\frac{\lfloor n}{q}$.

$$\lim_{\delta \doteq 0} \sin \lfloor n \pi (\pm\delta + \tfrac{p}{q}) \sin\left(\frac{1}{\sin \pi \lfloor n (\pm\delta + \frac{p}{q})}\right) -$$

$$\sin k\pi \sin\left(\frac{1}{\sin k\pi}\right) = \lim_{\delta \doteq 0} (-1)^k \sin \lfloor n \pi (\pm\delta) \cdot$$

$$\sin\left(\frac{1}{\sin \lfloor n \pi (\pm\delta)}\right) - 0 \cdot \sin \frac{1}{0}$$

$$= 0 \cdot \sin \infty - 0 \cdot \sin \infty$$

$$= 0,$$

and therefore the function is continuous. However, the function has no derivative at the point $x = \frac{p}{q}$; for

$$\frac{\lim_{\delta \doteq 0} \sin \lfloor n \pi (\pm\delta + \frac{p}{q}) \cdot \sin\left(\frac{1}{\sin \lfloor n \pi (\pm\delta + \frac{p}{q})}\right) - \sin k\pi \sin\left(\frac{1}{\sin k\pi}\right)}{\pm\delta} =$$

$$= \underset{\delta \doteq 0}{L} \left\{ \frac{(-1)^k \sin \lfloor \mu \pi(\pm\delta) \sin\left(\frac{(-1)^k}{\sin \lfloor \mu \pi(\pm\delta)}\right)}{\pm\delta} \right\}$$

$$= \underset{\delta \doteq 0}{L} \frac{\sin \lfloor \mu \pi(\pm\delta)}{\pm\delta} \cdot \underset{\delta \doteq 0}{L} \sin\left(\frac{1}{\sin \lfloor \mu \pi(\pm\delta)}\right).$$

But $\underset{\delta \doteq 0}{L} \dfrac{\sin \lfloor \mu \pi(\pm\delta)}{\pm\delta} = 1,$

and $\underset{\delta \doteq 0}{L} \sin\left(\dfrac{1}{\sin \lfloor \mu \pi(\pm\delta)}\right) = \sin(\pm\infty).$

$\therefore \underset{\delta \doteq 0}{L} \dfrac{f(x+\delta) - f(x)}{\pm\delta} = \sin(\pm\infty).$

The $\sin(\pm\infty)$ oscillates between $+1$ and $-1$, and the derivative of the function does not exist.

<u>§3. Continuous functions having no derivative at an infinite number of points.</u> We shall construct the desired functions by aid of the condensation theorem, that is by the condensation of points which have no derivatives.

Let $\varphi(y)$ be a finite and continuous function for $-1 \leqq y \leqq +1$, and let it have at every point, except $y=0$, a definite finite derivative which is less than some finite

number $1$. Further let $\varphi(y) = 0$ for $y = 0$. First, we shall consider the case when $\varphi(y)$, for $y = 0$, has no definite derivative, but the derivative never exceeds a finite limit. Then $f(x)$, defined by the relation

$$f(x) = \sum_{n=1}^{\infty} \frac{\varphi(\sin \underline{|n} \pi x)}{(\underline{|n})^2},$$

is a finite and continuous function; for

$$-1 \leqq \sin \underline{|n} \pi x \leqq +1$$

and the series is uniformly convergent since its terms are not numerically greater than those of the convergent series

$$\sum_{n=1}^{\infty} \frac{F}{(\underline{|n})^2},$$

where $F$ is the upper limit of $\varphi(y)$.

We shall first consider the question of a derivative at the irrational points. For such values we always have,

$$\sin \underline{|n} \pi x \neq 0,$$

and, therefore according to hypothesis,

$$f(x) = \sum_{n=1}^{\infty} \frac{\varphi(\sin \underline{|n} \pi x)}{(\underline{|n})^2}$$

will have for irrational values of $x$

a finite, determinate derivative less than $f$. This follows because the series of the derivatives of the various terms, viz: $\pi \sum_{n=1}^{\infty} \frac{\varphi'(\sin \lfloor n \pi x)}{\lfloor n} \cos \lfloor n \pi x$, is uniformly convergent, since its terms are numerically less than those of the convergent series $\pi \sum_{n=1}^{\infty} \frac{F}{\lfloor n}$.

We shall now examine the derivative of $f(x)$ at rational points. Let $x = \frac{p}{q}$, where $p$ and $q$ are integers prime to each other. Let $m$ be a number such that $\lfloor m$ does not contain $\frac{q}{p}$ an integral number of times, but that $\lfloor m+1$ contains $\frac{q}{p}$, $k$ times. Then we have

$$\lim_{\delta \doteq 0} \frac{f\left(\frac{p}{q} \pm \delta\right) - f\left(\frac{p}{q}\right)}{\delta} = \pi \sum_{n=1}^{m} \frac{\varphi'\left(\sin \lfloor n \pi \frac{p}{q}\right)}{\lfloor n} \cos \lfloor n \pi \frac{p}{q}$$

$$+ \lim_{\delta \doteq 0} \sum_{n=m+1}^{\infty} \frac{\varphi(\pm \sin \lfloor n \pi \delta)}{(\lfloor n)^2 \cdot \pm \delta}.$$

This is true since $\sin \lfloor n \frac{p}{q}$ is not zero for any term in the first summation of the right hand member, and the same reasoning used for irrational values holds good in it.

It is sufficient then to examine the second summation $\sum_{n=m+1}^{\infty} \frac{\varphi(\pm \sin \lfloor n \pi \delta)}{\pm \delta (\lfloor n)^2}$.

The sign of $\sin \lfloor n \pi \delta$ will be positive if $k$ is even and negative if $k$ is odd.

(1) Let $k$ be an even number then we have:

$$\underset{\delta \doteq 0}{L} \frac{f(\frac{p}{q}+\delta)-f(\frac{p}{q})}{\delta} = \pi \sum_{n=1}^{m} \frac{\varphi'(\sin \lfloor n \pi \frac{p}{q})}{\lfloor n} \cos \lfloor n \pi \frac{p}{q}$$

$$+ \underset{\delta \doteq 0}{L} \sum_{n=m+1}^{\infty} \frac{\varphi(+\sin \lfloor n \pi \delta)}{\delta (\lfloor n)^2}.$$

$$\underset{\delta \doteq 0}{L} \frac{f(\frac{p}{q}-\delta)-f(\frac{p}{q})}{-\delta} = \pi \sum_{n=1}^{m} \frac{\varphi'(\sin \lfloor n \pi \frac{p}{q})}{\lfloor n} \cos \lfloor n \pi \frac{p}{q}$$

$$+ \underset{\delta \doteq 0}{L} \sum_{n=m+1}^{\infty} \frac{\varphi(-\sin \lfloor n \pi \delta)}{\delta (\lfloor n)^2}.$$

In this case we see at once that the existence or non-existence of a derivative of $f(x)$ at any rational point $x = \frac{p}{q}$, will be the same as that of $\varphi(y)$, at the point $y = 0$, and the derivative will correspond in every respect to the derivative of $\varphi(y)$ at $y = 0$.

(2) Let $k$ be odd and $n$ odd. Then we have:

$$\underset{\delta \doteq 0}{L} \frac{f(\frac{p}{q}+\delta)-f(\frac{p}{q})}{\delta} = \pi \sum_{n=1}^{m} \frac{\varphi'(\sin \lfloor n \pi \frac{p}{q})}{\lfloor n} \cos \lfloor n \pi \frac{p}{q}$$

$$+\frac{L}{\delta \doteq 0}\left(\frac{1}{\lfloor m+1}\right)^2 \frac{\varphi(-\sin\lfloor n\,\pi\delta)}{\delta} + \frac{L}{\delta\doteq 0}\sum_{n=m+2}^{\infty}\frac{\varphi(\sin\lfloor n\,\pi\delta)}{\delta(\lfloor n)^2}$$

$$\frac{L}{\delta\doteq 0}\frac{f(\frac{p}{q}-\delta)-f(\frac{p}{q})}{-\delta} = \pi\sum_{n=1}^{m}\frac{\varphi'(\sin\lfloor n\,\pi\frac{p}{q})}{\lfloor n}\cos\lfloor n\,\pi\frac{p}{q}$$

$$+\frac{L}{\delta\doteq 0}\left(\frac{1}{\lfloor m+1}\right)^2\frac{\varphi(+\sin\lfloor n\,\pi\delta)}{\delta} + \frac{L}{\delta\doteq 0}\sum_{n=m+2}^{\infty}\frac{\varphi(-\sin\lfloor n\,\pi\delta)}{(\lfloor n)^2\delta}.$$

There will be no more terms in $-\sin\lfloor n\,\pi\delta$; for if $\frac{q}{p}$ is contained in $\lfloor m+1$, $k$ times, it will be contained in $\lfloor m+2$, $(m+2)k$ times; and since $n=m+1$ is odd this latter product is even and we have for all the rest of our terms $\frac{L}{\delta\doteq 0}\varphi(+\sin\lfloor n\,\pi\delta)$.

(3). Let $k$ be odd and $n$ even. Then we have:

$$\frac{L}{\delta\doteq 0}\frac{f(\frac{p}{q}+\delta)-f(\frac{p}{q})}{\delta} = \sum_{n=1}^{m}\frac{\varphi'(\sin\lfloor n\,\pi\frac{p}{q})}{\lfloor n}\cos\lfloor n\,\pi\frac{p}{q}$$

$$\frac{L}{\delta\doteq 0}\left\{+\left[\frac{1}{(\lfloor m+1)^2}+\frac{1}{(\lfloor m+2)^2}\right]\frac{\varphi(-\sin\lfloor n\,\pi\delta)}{\delta}+\sum_{n=m+3}^{\infty}\frac{\varphi(+\sin\lfloor n\,\pi\delta)}{(\lfloor n)^2}\right\}$$

$$\frac{L}{\delta\doteq 0}\frac{f(\frac{p}{q}-\delta)-f(\frac{p}{q})}{-\delta} = \sum_{n=1}^{m}\frac{\varphi'(\sin\lfloor n\,\pi\frac{p}{q})}{\lfloor n}\cos\lfloor n\,\pi\frac{p}{q}$$

$$\frac{L}{\delta\doteq 0}\left\{+\left[\frac{1}{(\lfloor m+1)^2}+\frac{1}{(\lfloor m+2)^2}\right]\frac{\varphi(+\sin\lfloor n\,\pi\delta)}{\delta}+\sum_{n=m+3}^{\infty}\frac{\varphi(-\sin\lfloor n\,\pi\delta)}{(\lfloor n)^2}\right\}$$

For, if $\frac{q}{p}$ is contained in $\lfloor m+1$, $k$ times, it is contained in $\lfloor m+2$, $(m+2)k$ times, and since these terms are odd they both give $\varphi(-\sin\lfloor n\,\pi\delta)$; but $\frac{q}{p}$ is con-

tained in $\underline{|m+3}$, $(m+3)(m+2)k$ times. Since $n = m+1$ is even, $(m+3)$ is even and the product $(m+3)(m+2)k$ will be even and thus give $\varphi(+\sin \underline{|n}\,\pi\delta)$. Likewise every term after this will have an even factor and thus give $\varphi(+\sin \underline{|n}\,\pi\delta)$.

Thus we see that for $k$ odd odd, $\varphi(+\sin \underline{|n}\,\pi\delta)$ and $\varphi(-\sin \underline{|n}\,\pi\delta)$ enter into the equation for both positive and negative approaches to zero; hence if either $\varphi(+\sin \underline{|n}\,\pi\delta)$ or $\varphi(-\sin \underline{|n}\,\pi\delta)$ oscillate between finite limits, the derivative of $f(x)$, at the point $x = \frac{p}{q}$, will oscillate on both sides of $x = \frac{p}{q}$.

Then if $\varphi(y)$ is a finite and continuous function for, $-1 \leqq y \leqq +1$, and has at every point except $y=0$, a definite, finite derivative, but at $y=0$ the derivative is indeterminate but remains between finite limits, we can make the following statements with regard to $f(x)$. $f(x)$ will have a definite, finite derivative at all irrational points. At all rational

points, for which $k$ is even, $f(x)$ will present the same characteristics with respect to its derivative as $\varphi(y)$ at the point $y=0$. At all rational points, for which $k$ is odd, $f(x)$ will have no definite derivative on either side of the point, if $\varphi(y)$ has no definite, finite derivative on one or both sides of $y=0$

Let $\varphi(y)$ be a finite and continuous function for $-1 \leqq y \leqq +1$, and let it have a definite, finite derivative for every point, except $y=0$, but as $y$ approaches indefinitely near to zero, let the derivative, in certain points, become greater than any definite finite number $g$.

Then at the irrational points, providing $f(x)$ is defined as at the beginning of the section, namely

$$f(x) = \sum_{n=1}^{\infty} \frac{\varphi(\sin \underline{n|}\, \pi x)}{(\underline{n|})^2},$$

then the existence of the derivative of $f(x)$ can no longer be shown. For $\underline{n|}\, x$, as $n$ increases indefinitely, passes

through values is to integral numbers as we wish, and as $\underline{|\mu}\,x$ approaches an integral number $\sin \underline{|\mu}\,\pi x$ approaches zero and the derivative of $\varphi(\sin \underline{|\mu}\,\pi x)$ is no longer less than a given finite value $g$.

At rational points, however, the function

$$f(x) = \sum_{\mu=1}^{\infty} \frac{\varphi(\sin \underline{|\mu}\,\pi \frac{p}{q})}{(\underline{|\mu})^2},$$

if the derivative of $\varphi(y)$ at $y=0$ is finite, although as $y$ approaches zero it becomes greater than any finite number $g$, will have a definite derivative

$$\pi \sum_{\mu=1}^{\infty} \frac{\varphi'(\sin \underline{|\mu}\,\pi \frac{p}{q})}{\underline{|\mu}} \cdot \cos \underline{|\mu}\,\frac{p}{q},$$

provided we show that:

$$\sum_{\mu=1}^{m} \left[ \frac{\varphi(\sin \underline{|\mu}\,\pi(\frac{p}{q}+\delta)) - \varphi(\sin \underline{|\mu}\,\pi \frac{p}{q})}{(\underline{|\mu})^2\,\delta} - \pi \frac{\varphi'(\sin \underline{|\mu}\,\pi \frac{p}{q})}{\underline{|\mu}} \cos \underline{|\mu}\,\pi \frac{p}{q} \right]$$

and $\dfrac{R_m(\frac{p}{q})}{\delta} \equiv \displaystyle\sum_{\mu=m+1}^{\infty} \frac{\varphi(\sin \underline{|\mu}\,\pi \frac{p}{q})}{(\underline{|\mu})^2\,\delta}$.

and $\dfrac{R_m(\frac{p}{q}+\delta)}{\delta} \equiv \displaystyle\sum_{\mu=m+1}^{\infty} \frac{\varphi(\sin \underline{|\mu}\,\pi(\frac{p}{q}+\delta))}{(\underline{|\mu})^2\,\delta}$

can each be made less than some arbitrarily small positive number $\sigma$, by taking $\delta$ sufficiently small.*

* Dini's Function Theorie § 103.

Let $m$ be the first number for which

$$\lfloor m > \frac{1}{|\delta|^{q}} \qquad (q > \tfrac{1}{2}),$$

then $$(\lfloor m)^2 > \frac{1}{|\delta|^{2q}}$$

$$\frac{1}{(\lfloor m)^2} < |\delta|^{2q}.$$

Furthermore let $g$ be the maximum value of $\varphi(y)$, for $-1 \leqq y \leqq +1$. Then we can write

$$R_m\left(\frac{p}{q}+\delta\right) < g \sum_{n=m+1}^{\infty} \frac{1}{(\lfloor n)^2},$$

and since in the series $1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots$ each term is greater than the sum of all the succeeding terms, we can put

$$R_m\left(\frac{p}{q}+\delta\right) < g \frac{1}{(\lfloor m)^2}.$$

Replacing $\frac{1}{(\lfloor m)^2}$ by its value in terms of $\delta$, we have

$$R_m\left(\frac{p}{q}+\delta\right) < g\, |\delta|^{2q}$$

and $$\frac{R_m\left(\frac{p}{q}+\delta\right)}{\delta} < g\,|\delta|^{2q-1}.$$

In a similar manner, we may prove

$$\frac{R_m\left(\frac{p}{q}\right)}{\delta} < g\,|\delta|^{2q-1}.$$

Since $q > \frac{1}{2}$, the exponent $2q-1$ will

always be positive, and we can make the right hand members of the last two equations as small as we choose by making $\delta$ sufficiently small. Hence two of our conditions, namely:

$$\frac{R_m\left(\frac{p}{q}+\delta\right)}{\delta} < \sigma_1$$

$$\text{and } \frac{R_m\left(\frac{p}{q}\right)}{\delta} < \sigma_2$$

are fulfilled.

Now since $m$ is finite, and $\varphi(y)$ has a definite derivative at every point, although it becomes very large as $y$ approaches zero,

$$\sum_{n=1}^{m}\left[\frac{\varphi(\sin \lfloor n \, \pi(\frac{p}{q}+\delta)) - \varphi(\sin \lfloor n \, \pi \frac{p}{q})}{(\lfloor n)^2 \delta} - \frac{\pi \varphi'(\sin \lfloor n \, \pi \frac{p}{q}) \cos \lfloor n \, \pi \frac{p}{q}}{\lfloor n}\right]$$

remains less than $\sigma$. All our conditions are now fulfilled and we can write:

$$\underset{\delta \doteq 0}{L} \frac{f(\frac{p}{q}+\delta) - f(\frac{p}{q})}{\delta} = \pi \sum_{n=1}^{m} \frac{\varphi'(\sin \lfloor n \, \pi \frac{p}{q}) \cos \lfloor n \, \pi \frac{p}{q}}{\lfloor n}$$

$$+ \underset{\delta \doteq 0}{L} \sum_{n=m+1}^{\infty} \frac{\varphi(\pm \sin \lfloor n \, \pi \, \delta)}{(\lfloor n)^2 \delta},$$

where $m$ is defined as on page 27, par. 2.

The character of the derivative will thus depend entirely on the second summation of the right hand member,

for $\sin \lfloor n \pi \frac{p}{q} \neq 0$ in the first term and by our hypothesis $\varphi'(\sin \lfloor n \pi \frac{p}{q})$ will exist and have at such points a definite value.

We have the following cases.

(1) When $k$ is even:

$$\underset{\delta \doteq 0}{L} \frac{f(\frac{p}{q}+\delta)-f(\frac{p}{q})}{\delta} = \pi \sum_{n=1}^{m} \frac{\varphi'(\sin \lfloor n \pi \frac{p}{q})}{\lfloor n} \cos \lfloor n \pi \frac{p}{q}$$

$$+ \underset{\delta \doteq 0}{L} \sum_{n=m+1}^{\infty} \frac{\varphi(+\sin \lfloor n \pi \delta)}{(\lfloor n)^2 \delta}.$$

$$\underset{\delta \doteq 0}{L} \frac{f(\frac{p}{q}-\delta)-f(\frac{p}{q})}{-\delta} = \sum_{n=1}^{m} \frac{\varphi'(\sin \lfloor n \pi \frac{p}{q})}{\lfloor n} \cos \lfloor n \pi \frac{p}{q}$$

$$+ \underset{\delta \doteq 0}{L} \sum_{n=m+1}^{\infty} \frac{\varphi(-\sin \lfloor n \pi \delta)}{(\lfloor n)^2 \delta}. *$$

(2) When $k$ is odd and $n$ is odd.

$$\underset{\delta \doteq 0}{L} \frac{f(\frac{p}{q}+\delta)-f(\frac{p}{q})}{\delta} = \sum_{n=1}^{m} \frac{\varphi'(\sin \lfloor n \pi \frac{p}{q})}{\lfloor n} \cos \lfloor n \pi \frac{p}{q}$$

$$+ \underset{\delta \doteq 0}{L} \frac{1}{(\lfloor m+1)^2} \frac{\varphi(-\sin \lfloor n \pi \delta)}{\delta} + \underset{\delta \doteq 0}{L} \sum_{n=m+2}^{\infty} \frac{\varphi(+\sin \lfloor n \pi \delta)}{(\lfloor n)^2 \delta}$$

$$\underset{\delta \doteq 0}{L} \frac{f(\frac{p}{q}-\delta)-f(\frac{p}{q})}{-\delta} = \sum_{n=1}^{m} \frac{\varphi'(\sin \lfloor n \pi \frac{p}{q})}{\lfloor n} \cos \lfloor n \pi \frac{p}{q}$$

$$+ \underset{\delta \doteq 0}{L} \frac{1}{(\lfloor m+1)^2} \frac{\varphi(+\sin \lfloor n \pi \delta)}{\delta} + \underset{\delta \doteq 0}{L} \sum_{n=m+2}^{\infty} \frac{\varphi(-\sin \lfloor n \pi \delta)}{(\lfloor n)^2 \delta} \; **$$

(3) When $k$ is odd and $n$ is even:

$$\underset{\delta \doteq 0}{L} \frac{f(\frac{p}{q}+\delta)-f(\frac{p}{q})}{\delta} = \sum_{n=1}^{m} \frac{\varphi'(\sin \lfloor n \pi \frac{p}{q})}{\lfloor n} \cos \lfloor n \pi \frac{p}{q}$$

* see page 28 (1). ** see page 28 (2)

$$\lim_{\delta \doteq 0}\left[\frac{1}{((m+1)!)^2}+\frac{1}{((m+2)!)^2}\right]\frac{\varphi(-\sin n!\,\pi\,\delta)}{\delta} + \lim_{\delta \doteq 0}\sum_{n=m+3}^{\infty}\frac{\varphi(+\sin n!\,\pi\,\delta)}{(n!)^2\,\delta}$$

$$\lim_{\delta \doteq 0}\frac{f(\frac{p}{q}-\delta)-f(\frac{p}{q})}{-\delta} = \sum_{n=1}^{m}\frac{\varphi'(\sin n!\,\pi\,\frac{p}{q})}{\delta}\cos n!\,\pi\,\frac{p}{q}$$

$$+\lim_{\delta \doteq 0}\left[\frac{1}{((m+1)!)^2}+\frac{1}{((m+2)!)^2}\right]\frac{\varphi(+\sin n!\,\pi\,\delta)}{\delta} + \lim_{\delta \doteq 0}\sum_{n=m+3}^{\infty}\frac{\varphi(-\sin n!\,\pi\,\delta)}{(n!)^2\,\delta} \quad *$$

Hence when $k$ is even we see from (1) that the derivative of $f(x)$ corresponds in every way to the derivative of $\varphi(y)$ at $y=0$. When $k$ is odd we see from (2) and (3) that, if $\varphi'(y)$ oscillates on either side of $y=0$, the derivative of $f(x)$ oscillates on both sides of the point $x=\frac{p}{q}$. If however $\varphi'(y)$ has the limit $+\infty$ on one side of $y=0$ and $-\infty$ on the other, the derivative of $f(x)$ will have the limit $-\infty$ on the first side and $+\infty$ on the second. This follows from the fact that

$$\frac{1}{((m+1)!)^2} > \sum_{n=m+2}^{\infty}\frac{1}{(n!)^2}$$

$$\text{and } \frac{1}{((m+1)!)^2}+\frac{1}{((m+2)!)^2} > \sum_{n=m+3}^{\infty}\frac{1}{(n!)^2},$$

and the sign of the second term determines the sign of the right hand member of the expressions for the derivatives.

We may therefore sum up our

see page 29 (3).

results as follows: Given $\varphi(y)$, finite and continuous, for $-1 \leqq y \leqq +1$, and having a definite derivative at every point except $y = 0$, we can say:

(a) If the derivative of $\varphi(y)$ oscillates between infinite limits at the point $y=0$, $f(x)$ has no definite derivative at the points $x = \frac{k}{q}$, that is the rational points.

(b) If the derivative of $\varphi(y)$ is $+\infty$ on one side of $y=0$ and $-\infty$ on the other, the derivative of $f(x)$ has the same characteristics at the rational points, for which $k$ is even. At the rational points for which $k$ is odd the limits on the two sides of the point are interchanged.

(c) The existence or non-existence of the derivative of $f(x)$ at the irrational points is wholly uncertain

Example 1: Given the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin \lfloor n \, \pi x \sin\left(\frac{1}{\sin \lfloor n \, \pi x}\right)}{(\lfloor n)^2} .$$

The simple function corresponding to this is

$$\varphi(y) = y \sin \frac{1}{y} .$$

$\varphi(y)$ is continuous in the interval $-1 \leq y \leq +1$, for

$$\underset{\delta \doteq 0}{L} \; (y_0 \pm \delta) \sin\left(\frac{1}{y_0 \pm \delta}\right) - y_0 \sin\left(\frac{1}{y_0}\right) =$$

$$\underset{\delta \doteq 0}{L} \pm \delta \sin\left(\frac{1}{y_0 \pm \delta}\right) + \underset{\delta \doteq 0}{L}\left(y_0 \sin\left(\frac{1}{y_0 \pm \delta}\right) - y_0 \sin\frac{1}{y_0}\right)$$

$$= 0.$$

However $\varphi(y)$ has no definite derivative at the point $y = 0$, but its derivative oscillates between $+1$ and $-1$, as shown in Ex. 1, page 19.

Therefore from the results obtained on page 30, we can say:

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin \lfloor n \, \pi x \, \sin\left(\frac{1}{\sin \lfloor n \, \pi x}\right)}{(\lfloor n)^2}$$

has at every irrational point a definite, finite derivative, which is

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{\sin \lfloor n \, \pi x}\right) - \frac{\cos\left(\frac{1}{\sin \lfloor n \, \pi x}\right)}{\sin \lfloor n \, \pi x}}{\lfloor n} \cdot \cos \lfloor n \, \pi x.$$

At every irrational point the derivative of $f(x)$ will oscillate between finite limits on both sides of the point.

Example 2: Given the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(\sin \lfloor n \, \pi x)^{2/3}}{(\lfloor n)^2}.$$

The simple function corresponding

to this is

$$\varphi(y) = y^{2/3}.$$

$\varphi(y)$ is continuous in the interval $- \leqq y \leqq +1$; for putting

$$z = (y_0 \pm \delta)^{2/3},$$
$$z^3 = (y_0 \pm \delta)^2$$
$$= y_0^2 \pm 2 y_0 \delta + \delta^2$$
$$L_{\delta \doteq 0}\, z^3 = y_0^2$$
$$L_{\delta \doteq 0}\, z = y_0^{2/3}.$$
$$\therefore L_{\delta \doteq 0} (y_0 \pm \delta)^{2/3} - y_0^{2/3} = 0.$$

The derivative of $\varphi(y)$ at the point $y = 0$ is $+\infty$ on the right and $-\infty$ on the left of the point, as was shown in Ex. 3, page 20.

Therefore from the results obtained on page 37, we can say: We know nothing whatever of the derivative of

$$f(x) = \sum_{n=1}^{\infty} \frac{(\sin \lfloor n \, \pi x)^{2/3}}{(\lfloor n)^2},$$

at irrational points, but at every rational point where $k$ is even the derivative is $+\infty$ on the right and $-\infty$ on the left. Where $k$ is odd the derivative is $-\infty$ on the right and $+\infty$ on the left.

Example 3: Given the function
$$f(x) = \sum_{n=1}^{\infty} \frac{|\sin \lfloor n \pi x|}{(\lfloor n)^2}.$$

The simple function corresponding to this is
$$\varphi(y) = |y|.$$
$\varphi(y)$ is continuous in the interval $-1 \leqq y \leqq +1$; for
$$\lim_{\delta \doteq 0} |x \pm \delta| - |x| = \lim_{\delta \doteq 0} (|x| \pm |\delta| - |x|)$$
$$= 0$$
The derivative of $\varphi(y)$ at the point $y = 0$ is $+1$ on the right and $-1$ on the left, as shown in Ex. 4, page 20.

Therefore from the results obtained on page 30, we can say:
$$f(x) = \sum_{n=1}^{\infty} \frac{|\sin \lfloor n \pi x|}{(\lfloor n)^2}$$
has at every irrational point a definite derivative which is,
$$\pi \sum_{n=1}^{\infty} \frac{\cos \lfloor n \pi x}{\lfloor n}.$$

At the rational points, for which $k$ is even, the derivative of $f(x)$ is positive on the right and negative on the left; but at the rational points for which $k$ is odd, the derivative

is negative on the right and positive on the left.

Example 4: Given the function
$$f(x) = \sum_{n=1}^{\infty} \frac{\sin n!\,\pi x \,\sin\left[\log(\sin n!\,\pi x)^2\right]}{(n!)^2}.$$

The simple function corresponding to this is.
$$\varphi(y) = y \sin(\log y^2).$$
$\varphi(y)$ is continuous in the interval $-1 \leqq y \leqq +1$; for
$$\underset{\delta=0}{L}\ (y_0 \pm \delta)\sin(\log[y_0 \pm \delta]^2) - y_0 \sin(\log y_0^2) =$$
$$\underset{\delta=0}{L}\ y_0 \sin(\log[y_0 \pm \delta]^2) - y_0 \sin(\log y_0^2) \pm \delta \sin(\log[y_0 \pm \delta]^2)$$
$$= 0 \cdot \sin(\log y_0^2)$$
$$= 0.$$
The derivative of $\varphi(y)$ at the point $y = 0$ oscillates between $+1$ and $-1$, on both sides of the point, as shown in Ex. 5, p. 21.

Therefore from the results given on page 15 we can say:
$$f(x) = \sum_{n=1}^{\infty} \frac{\sin n!\,\pi x \,\sin\left[\log(\sin n!\,\pi x)^2\right]}{(n!)^2}$$

has at every irrational point a definite finite derivative which is
$$\pi \sum_{n=1}^{\infty} \frac{\sin(\log x^2) + 2\cos(\log x^2)}{n!} \cdot \cos n!\,\pi x.$$

At every rational point the derivative of $f(x)$ will oscillate between finite limits on both sides of the point.

Example 5: Given the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\frac{1}{\log(\sin \lfloor n \pi x)^2}}{(\lfloor n)^2}.$$

The simple function corresponding to this is

$$\varphi(y) = \frac{1}{\log y^2}.$$

$\varphi(y)$ is continuous for the interval $-1 \leqq y \leqq +1$; for

$$\lim_{\delta=0} \frac{1}{\log(y_0 \pm \delta)^2} - \frac{1}{\log y_0^2} = \frac{1}{\log \lim_{\delta=0} (y_0+\delta)^2} - \frac{1}{\log y_0^2}$$

$$= \frac{1}{\log y_0^2} - \frac{1}{\log y_0^2}$$

$$= 0.$$

The derivative of $\varphi(y)$ at the point $y = 0$ is $+\infty$ on the right and $-\infty$ on the left.

Therefore from the results given on page 37, we can say: We know nothing whatever of the derivative of

$$f(x) = \sum_{n=1}^{\infty} \frac{\frac{1}{\log(\sin \lfloor n \pi x)^2}}{(\lfloor n)^2},$$

at the irrational points. At the

rational points where $k$ is even the derivative is $+\infty$ on the right and $-\infty$ on the left; where $k$ is odd the derivative is $-\infty$ on the right and $+\infty$ on the left.

Example 6: Given the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(\sin^2 \lfloor n \pi x)^{\frac{1-\alpha}{2}} \sin(\frac{1}{\sin \lfloor n \pi x})}{(\lfloor n)^2},$$

where $0 < \alpha < 1$. The simple function corresponding to this is

$$\varphi(y) = (y^2)^{\frac{1-\alpha}{2}} \sin \frac{1}{y}.$$

$\varphi(y)$ is continuous for the interval $-1 \leqq y \leqq +1$; for putting

$$z = ([y_0 \pm \delta]^2)^{\frac{1-\alpha}{2}},$$

then $z^{\frac{2}{1-\alpha}} = (y_0 \pm \delta)^2$

$$\underset{\delta \doteq 0}{L} z^{\frac{2}{1-\alpha}} = y_0^2.$$

$$\therefore \underset{\delta \doteq 0}{L} ([y_0 \pm \delta]^2)^{\frac{1-\alpha}{2}} \sin \frac{1}{(y_0 \pm \delta)} - (y_0^2)^{\frac{1-\alpha}{2}} \sin \frac{1}{y_0} =$$

$$(y_0^2)^{\frac{1-\alpha}{2}} \sin \frac{1}{\underset{\delta \doteq 0}{L} (y_0 \pm \delta)} - (y_0^2)^{\frac{1-\alpha}{2}} \sin \frac{1}{y_0} = 0.$$

$\varphi(y)$ has no derivative at the point $y=0$; for

$$\underset{\delta \doteq 0}{L} \frac{[(0 \pm \delta)]^{\frac{1-\alpha}{2}} \sin(\frac{1}{0 \pm \delta}) - 0 . \frac{1}{\sin 0}}{\pm \delta} = \underset{\delta \doteq 0}{L} \frac{\delta^{1-\alpha} \sin \frac{1}{\pm \delta}}{\pm \delta}$$

$$= \underset{\delta \doteq 0}{L} \frac{\sin \frac{1}{\delta}}{\delta^{\alpha}}$$

$$= \frac{\sin \infty}{0},$$

which has no definite value but oscillates between $+\infty$ and $-\infty$.

Therefore from the results given on page 37, we can say: We know nothing whatever of the derivative of

$$f(x) = \sum_{n=1}^{\infty} \frac{(\sin^2 \underline{|n}\, \pi x)^{\frac{1-x}{2}} \sin\left(\frac{1}{\sin \underline{|n}\, \pi x}\right)}{(\underline{|n})^2}$$

at the irrational points. At the rational points the derivative oscillates between $+\infty$ and $-\infty$ on both sides of the point.

Example 7: Given the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{(\underline{|n})^2} \log \frac{(\underline{|n}\, x)^{E(\underline{|n}\, x)}}{E(\underline{|n}\, x)!},$$

where $E(\underline{|n}\, x)$ is the largest integer contained in $\underline{|n}\, x$. We know

$$\frac{1}{\underline{|n}} \log \frac{(\underline{|n}\, x)^{E(\underline{|n}\, x)}}{E(\underline{|n}\, x)!} = \frac{1}{\underline{|n}} [E(\underline{|n}\, x) \log \underline{|n}\, x - \log E(\underline{|n}\, x)!].$$

We may write

$$\underline{|n}\, x = E(\underline{|n}\, x) + \varepsilon,$$

where $0 < \varepsilon < 1$. Then we get

$$\frac{\underline{|n}\, x}{E(\underline{|n}\, x)} = 1 + \frac{\varepsilon}{E(\underline{|n}\, x)}$$

$$\frac{E(\underline{|n}\, x)}{\underline{|n}} = x - \frac{\varepsilon}{\underline{|n}}.$$

$$\therefore \underset{n \doteq \infty}{L} \frac{L^n x}{E(L^n x)} = 1$$

$$\text{and } \underset{n \doteq \infty}{L} \frac{E(L^n x)}{L^n} = x.$$

We may put

$$\frac{E(L^n x) \log (L^n x) - \log E(L^n x)!}{L^n} = \frac{1}{L^n}\Big\{ E(L^n x) \log E(L^n x)$$

$$- \log E(L^n x)! + E(L^n x) \log L^n x - E(L^n x) \log E(L^n x)\Big\}$$

$$= \frac{1}{L^n}\Big\{ E(L^n x) \log E(L^n x) - \log E(L^n x)! + E(L^n x) \log \frac{L^n x}{E(L^n x)}$$

$$= \frac{E(L^n x)}{L^n}\Big\{ E(L^n x) \log E(L^n x) - \log E(L^n x)! + \log \frac{L^n x}{E(L^n x)}$$

$$\therefore \underset{n \doteq \infty}{L} \frac{E(L^n x) \log L^n x - \log E(L^n x)!}{L^n} = \underset{n \doteq \infty}{L} \frac{E(L^n x)}{L^n}\Big\{ \frac{E(L^n x) \log E(L^n x)}{E(L^n x)}$$

$$\ldots - \frac{\log E(L^n x)!}{E(L^n x)} + \log \frac{L^n x}{E(L^n x)} \Big\}$$

We know that

$$\underset{y \doteq \infty}{L} \frac{1}{y} \sqrt[y]{y!} = \frac{1}{e} \;*.$$

Hence by taking the logarithm, we have:

$$\underset{y \doteq \infty}{L} \left[ \frac{1}{y} \log (y!) - \log y \right] = -1$$

$$\underset{y \doteq \infty}{L} \left( \frac{y \log y - \log (y!)}{y} \right) = 1.$$

Substituting $y = E(L^n x)$ in the preceding equation, we have:

* Esercizi di Calcolo Infinitesimale § 11.

$$\underset{n \doteq \infty}{L} \left\{ \frac{E(\underline{|n}\, x) \log E(\underline{|n}\, x) - \log E(\underline{|n}\, x)!}{E(\underline{|n}\, x)} \right\} = 1 .$$

Hence substituting the values in the limit of the terms of the right hand member of the fourth equation on page 45, we have:

$$\underset{n \doteq \infty}{L} \quad \frac{E(\underline{|n}\, x) \log (\underline{|n}\, x) - \log E(\underline{|n}\, x)!}{\underline{|n}} = x(1+0)$$

$$= x .$$

$$\therefore \frac{1}{\underline{|n}} \log \frac{(\underline{|n}\, x)^{E(\underline{|n}\, x)}}{E(\underline{|n}\, x)!} = x .$$

Then if $A$ is the maximal value of $x$ we can put:

$$f(x) \leqq \sum_{n=1}^{\infty} \frac{A}{\underline{|n}} .$$

Therefore since the terms of $f(x)$ are less or at most equal to the terms of the uniformly convergent series $\sum_{n=1}^{\infty} \frac{A}{\underline{|n}}$,

we can say that $f(x)$ is a uniformly convergent series, and therefore a continuous function.

We shall now examine the derivative of $f(x)$.

$$\frac{f(x+h) - f(x)}{h} = \sum_{n=1}^{\infty} \frac{1}{\underline{|n}} \left\{ \frac{1}{\underline{|n}\, h} \log \frac{(\underline{|n}\, x + \underline{|n}\, h)^{E(\underline{|n}\, x + \underline{|n}\, h)}}{E(\underline{|n}\, x + \underline{|n}\, h)!} \right.$$

$$\left. - \frac{1}{\underline{|n}\, h} \log \frac{(\underline{|n}\, x)^{E(\underline{|n}\, x)}}{E(\underline{|n}\, x)!} \right\} ,$$

and the limit of the quantity in brackets, $n \doteq \infty$, becomes $\frac{x \pm h}{h} - \frac{x}{h}$. This follows directly from the third equation page 46. Since the expression in the brackets has a finite limit for $n = \infty$, we can always find a finite number $A'$ which is greater than or equal to it for any values of $n$ and $h$. Then since the terms of $\frac{f(x \pm h) - f(x)}{\pm h}$ will always be less or equal to the terms of the absolutely convergent series $\sum_{n=1}^{\infty} \frac{A'}{\lfloor n}$, we can write

$$\frac{f(x \pm h) - f(x)}{\pm h} \leqq \frac{A'}{\lfloor n}.$$

Since this equation is true for every value of $h$, we may conclude that it is true for the limit $h \doteq 0$, and that $f(x)$ always has a derivative right and a derivative left of the point $x$.

We shall now show that the derivative right of the point and the derivative left of the point are unequal for rational points, and therefore that the derivative does not

exist.

Let $x = \frac{p}{q}$ be any rational point. Let $m$ be a number such $|\underline{m+1}$ contains $q$, but that $|\underline{m}$ does not contain $q$ an integral number of times.

$$\frac{f(\frac{p}{q}+h)-f(\frac{p}{q})}{h} = \sum_{n=1}^{m} \frac{1}{(|\underline{n})^2 h}\Big\{E(|\underline{n}\tfrac{p}{q}+|\underline{n}h)\log(|\underline{n}\tfrac{p}{q}+|\underline{n}h)$$

$$-E(|\underline{n}\tfrac{p}{q})\log(|\underline{n}\tfrac{p}{q}) + \log E(|\underline{n}\tfrac{p}{q})! - \log E(|\underline{n}\tfrac{p}{q}+|\underline{n}h)!\Big\}$$

$$+\sum_{n=m+1}^{\infty} \frac{1}{(|\underline{n})^2 h}\Big\{E(|\underline{n}\tfrac{p}{q}+|\underline{n}h)\log(|\underline{n}\tfrac{p}{q}+|\underline{n}h)$$

$$-E(|\underline{n}\tfrac{p}{q})\log(|\underline{n}\tfrac{p}{q}) + \log E(|\underline{n}\tfrac{p}{q})! - \log E(|\underline{n}\tfrac{p}{q}+|\underline{n}h)!\Big\}$$

If now we consider only the second summation of the right hand member and replace $|\underline{n}$ by $|\underline{n}\,q$, we have:

$$\frac{1}{q^2}\sum_{n=\frac{m+1}{q}}^{\infty} \frac{1}{(|\underline{n})^2}\Big\{E(|\underline{n}p+|\underline{n}qh)\log(|\underline{n}p+|\underline{n}qh)$$

$$-E(|\underline{n}p)\log(|\underline{n}p)+\log E(|\underline{n}p)! - \log E(|\underline{n}p+|\underline{n}qh)!\Big\}$$

For any finite $n$ we can always take $h$ so small that $|\underline{n}\,qh<1$. Then for positive $h$ we have

$$E(|\underline{n}p+|\underline{n}qh) = E(|\underline{n}p) = |\underline{n}p.$$

For $(|\underline{n}p+|\underline{n}qh)$ can never become as large as $|\underline{n}+1$. For negative $h$ we have

$$E(|\underline{n}p+|\underline{n}qh) = E(|\underline{n}p)-1 = |\underline{n}p-1.$$

For $(|\underline{n}p+|\underline{n}qh)$ will always be less

than $\lfloor\mu\, p$, and the largest integer contained in it will be $\lfloor\mu\, p - 1$. Then if we consider any term of this series, we have for positive $h$ by substituting

$$E(\lfloor\mu\, p + \lfloor\mu\, q\, h) = \lfloor\mu\, p;$$

$$\frac{1}{(\lfloor\mu)^2}\left\{\lfloor\mu\, p \cdot \frac{\log(\lfloor\mu\, p + \lfloor\mu\, q\, h) - \log(\lfloor\mu\, p)}{h}\right\} = \frac{1}{(\lfloor\mu)^2}\left(\lfloor\mu\, p \log\left(\frac{\lfloor\mu\, p + \lfloor\mu\, q\, h}{\lfloor\mu\, p}\right)^{\frac{1}{h}}\right)$$

$$= \frac{1}{(\lfloor\mu)^2}\left[\lfloor\mu\, p \log\left(1 + \frac{q}{p}h\right)^{\frac{1}{h}}\right]$$

$$= \frac{1}{(\lfloor\mu)^2}\left[\lfloor\mu\, p \log\left(1 + \frac{q}{p} + \frac{1-h}{\lfloor 2}\frac{q^2}{p^2} + \frac{(1-h)(1-2h)}{\lfloor 3}\frac{q^3}{p^3} + \frac{(1-h)(1-2h)(1-3h)}{\lfloor 4}\frac{q^4}{p^4} + \cdots\right)\right]$$

which in the limit $h \doteq 0$ becomes

$$= \frac{1}{(\lfloor\mu)^2}\left[\lfloor\mu\, p \log\left(1 + \frac{q}{p} + \frac{1}{\lfloor 2}\frac{q^2}{p^2} + \frac{1}{\lfloor 3}\frac{q^3}{p^3} + \cdots\right)\right]$$

$$= \frac{1}{(\lfloor\mu)^2}\left[\lfloor\mu\, p \log\left(e^{\frac{q}{p}}\right)\right]$$

$$= \frac{1}{(\lfloor\mu)^2}\left[\lfloor\mu\, p \cdot \frac{q}{p}\right]$$

$$\doteq \frac{q}{\lfloor\mu}.$$

For the same term of the series for negative $h$, by putting $E(\lfloor\mu\, p + \lfloor\mu\, q\, h) = \lfloor\mu\, p - 1$, we have:

$$\frac{1}{(\lfloor\mu)^2}\left[(\lfloor\mu\, p - 1)\,\frac{\log(\lfloor\mu\, p + \lfloor\mu\, q\, h) - \log(\lfloor\mu\, p)}{h}\right.,$$

which in the limit $h=0$, by a process exactly similar to that for positive $h$ becomes

$$= \frac{(\lfloor n \, p - 1)\lfloor n \, q}{(\lfloor n)^2 \lfloor n \, p} = \frac{\lfloor n \, q}{(\lfloor n)^2} - \frac{1}{(\lfloor n)^2}\frac{q}{p}.$$

Then we can write for the series for positive $h$

$$\frac{1}{q}\sum_{n=m+1}^{\infty}\frac{1}{\lfloor n}.$$

For negative $h$, we have

$$\frac{1}{q}\sum_{n=m+1}^{\infty}\frac{1}{\lfloor n} - \frac{1}{q}\sum_{n=m+1}^{\infty}\frac{1}{(\lfloor n)^2}.$$

The two limits are unequal.

For the series which is a summation from $n=1$ to $n=\infty$ the two limits are equal: for, since $\lfloor n \, x$ is not an integral number, by taking $h$ sufficiently small we shall have for both positive and negative $h$:

$$E(\lfloor n \, x + \lfloor n \, h) = E(\lfloor n \, x).$$

Then the derivative right and the derivative left of any rational point will both be the sum of two terms. One term in the one will be equal to one term in the other, but the other two terms will be unequal and therefore the sums unequal, and the function has

at no rational point a derivative.

For irrational values of $x$, $\underline{n} x$ will not be an integer and for both positive and negative $h$ we shall have

$$E(\underline{n} x + \underline{n} h) = E(\underline{n} x).$$

Therefore the limits of the derivative for positive and negative $h$ will be the same, and the function $f(x)$ will have a derivative at every irrational point.

Example 8: Given the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\varphi(2^n x)}{2^n \cdot 2^n}.$$

to show that it is continuous but has no derivative at the points of the form $x' = \frac{m'}{2^m}$, where $m'$ is an integer and $m$ the least number which will make $x' 2^m$ an integer. Take the function

$$\varphi(x) = E(x) + \sqrt{x - E(x)},$$

where $E(x)$ represents the largest integer in $x$. The value of this function, when $n \leqq x < n+1$, is

$$\varphi(x) = n + \sqrt{x - n};$$

for in that case $n$ will be the largest integer in $E(x)$. When $x = n+1$, we have

$$\varphi(x) = n+1 + \sqrt{n+1-(n+1)}$$
$$= n+1$$
$$\therefore \varphi(n+1 \pm h) - \varphi(n+1) = n+1 + \sqrt{n+1 \pm h-(n+1)} - (n+1)$$
$$\varphi(n+1 \pm h) - \varphi(n+1) = \sqrt{\pm h}$$
$$\therefore \lim_{h \doteq 0} [\varphi(n+1 \pm h) - \varphi(n+1)] = \lim_{h \doteq 0} \sqrt{\pm h} = 0$$

For $n \leqq x < n+1$, we know on the other hand

$$\varphi(x \pm h) - \varphi(x) = n + \sqrt{x \pm h - n} - (n + \sqrt{x-n})$$
$$= \sqrt{x \pm h - n} - \sqrt{x-n}$$
$$= \sqrt{\pm h}$$
$$\therefore \lim_{h \doteq 0} [\varphi(x \pm h) - \varphi(x)] = \lim_{h \doteq 0} \sqrt{\pm h} = 0.$$

Therefore we can say that $\varphi(x)$ is continuous in the interval $n \leqq x \leqq n+1$, and since $n$ may be any integer, $\varphi(x)$ is continuous for any value of $x$.

We shall now consider the function

$$f(x) = \sum_{n=0}^{\infty} \frac{\varphi(2^n x)}{2^n 2^n}$$

as formed by multiplying the terms of the uniformly convergent series $x \sum_{n=0}^{\infty} \frac{1}{2^n}$ by the quantity $\frac{\varphi(2^n x)}{2^n x}$. We know that for integral values of $n$,

$$E(2^n x) = 2^n x.$$

Therefore for the same values,

$$\varphi(x) = E(2^n x) + \sqrt{2^n x - E(2^n x)}$$
$$= 2^n x$$
$$\therefore \lim_{n \doteq \infty} \frac{\varphi(2^n x)}{2^n x} = \lim_{n \doteq \infty} \frac{2^n x}{2^n x} = 1.$$

It follows then that $\sum_{n=0}^{\infty} \frac{\varphi(2^n x)}{2^n \cdot 2^n}$ is a uniformly convergent series, since it is equal to the product of a uniformly convergent series and a second factor equal to 1, when $n \doteq \infty$. We can therefore say that $f(x)$ is a continuous function. *

We shall now examine the derivative of $f(x)$, for those values of the form $x' = \frac{m'}{2^m}$. We may now write

$$\frac{f(x'+h) - f(x)}{h} = \sum_{n=0}^{\infty} \frac{\varphi(2^n x' + 2^n h) - \varphi(2^n x')}{2^n \cdot 2^n h}.$$

Since for a positive $h$, $\varphi(2^n x' + 2^n h) - \varphi(2^n x')$ is always positive, any single term will be less than the summation, and we may put

$$\frac{f(x'+h) - f(x')}{h} > \frac{\varphi(2^m x' + 2^m h) - \varphi(2^m x')}{2^m \cdot 2^m h}.$$

By taking $h$ sufficiently small, we can always make $2^m h < 1$, and since $2^m x' = m'$, we have

* Dini Functionen Theorie p 147.

$$\varphi(2^m x' + 2^m h) = E(m') + \sqrt{m' + 2^m h - E(m')}$$
$$= m' + \sqrt{2^m h}.$$

But $\varphi(2^m x') = \varphi(m') = m'$,
and we have from the preceding equation
$$\varphi(2^m x' + 2^m h) - \varphi(2^m x') = \sqrt{2^m h}.$$
$$\therefore \frac{f(x'+h) - f(x')}{h} > \sqrt{2^m h} \cdot \frac{1}{2^m \cdot 2^m h}$$
$$> \frac{1}{(2\sqrt{2})^m \sqrt{h}}.$$

Since as $h \doteq 0$ the right hand member of this equation increases beyond all limit, we can say that the derivative of $f(x)$ to the right of $(x')$ is infinite.

We shall now examine the derivative left of the point $x'$ and shall find that it is finite. Consider the quantity
$$\frac{\varphi(2^n x' - 2^n h) - \varphi(2^n x')}{-2^n h} \equiv u_n.$$

For $n < m$, $2^n x'$ is not an integer, and since we can always select $h$ so that $2^n h < 1$, we have
$$E(2^n x' - 2^n h) = E(2^n x')$$
$$\therefore \varphi(2^n x' - 2^n h) - \varphi(2^n x') = E(2^n x' - 2^n h) + \sqrt{2^n x' - 2^n h - E(2^n x' - 2^n h)}$$
$$- \left( E(2^n x') + \sqrt{2^n x' - E(2^n x')} \right)$$

$$= \sqrt{2^n x' - 2^n h - E(2^n x')} - \sqrt{2^n x' - E(2^n x')}.$$

We know

$$\underset{h \doteq 0}{L} \frac{\sqrt{N - 2^n h} - \sqrt{N}}{-h} = \underset{h \doteq 0}{L} \frac{2^{n-1}}{\sqrt{N - 2^n h}} = \frac{2^{n-1}}{\sqrt{N}} \quad *$$

Putting $N = 2^n x' - E(2^n x')$, we have

$$\underset{h \doteq 0}{L} \frac{\sqrt{2^n x' - E(2^n x') - 2^n h} - \sqrt{2^n x' - E(2^n x')}}{-h} = \frac{2^{n-1}}{\sqrt{2^n x' - E(2^n x')}}.$$

Substituting this value in last equation on page 54, we have

$$\underset{h \doteq 0}{L} [\varphi(2^n x' - 2^n h) - \varphi(2^n x') = \frac{2^{n-1}}{\sqrt{2^n x' - E(2^n x')}}.$$

$$\therefore \quad \underset{h \doteq 0}{L} \quad u_n = \frac{1}{2\sqrt{2^n x' - E(2^n x')}}.$$

Now since for $n < m$, $2^n x'$ is not an integer consequently $\sqrt{2^n x' - E(2^n x')}$ cannot be zero, and $u_n$ will always be finite.

We shall now examine $u_n$, when $n \geqq m$, and shall find $\underset{h \doteq 0}{L} u_n = \frac{1}{2}$. When $n \geqq m$, $2^n x'$ is an integer. Now if $k$ is the first integer greater than $2^n h$ we have

$$\varphi(2^n x' - 2^n h) - \varphi(2^n x') = 2^n x' - k + \sqrt{2^n x' - 2^n h - (2^n x' - k)}$$
$$- (2^n x' + \sqrt{2^n x' - 2^n x'}$$
$$= \sqrt{k - 2^n h} - k.$$

$$\therefore \quad u_n = \frac{\sqrt{k - 2^n h} - k}{-2^n h}.$$

If $2^n h > 1$, then we have

*Esercizi di Calcolo Infinitesimale p. 113.

$$k - \sqrt{k - 2^n h} \leqq h$$

$$\leqq 2^n h + 1.$$

$$\therefore \quad u_n < \frac{2^n h + 1}{2^n h}$$

$$< 1 + \frac{1}{2^n h}$$

$$< 2.$$

If $2^n h \leqq 1$, then

$$u_n = \frac{1 - \sqrt{1 - 2^n h}}{2^n h},$$

and by taking the derivative with respect to $2^n h$ of both numerator and denominator, we have

$$\lim_{2^n h = 0} u_n = \lim_{2^n h = 0} \frac{\frac{1}{2}\sqrt{1 - 2^n h}}{1}$$

$$= \frac{1}{2}.$$

We may therefore say that the derivative left of $x'$ is

$$\lim_{h=0} \frac{f(x'-h) - f(x')}{-h} = \lim_{h=0} \sum_{n=0}^{m-1} \frac{u_n}{2^n} + \lim_{h=0} \sum_{n=m}^{\infty} \frac{u_n}{2^n}$$

$$\leqq \sum_{n=0}^{m-1} \frac{1}{2^{n+1}} \cdot \frac{1}{\sqrt{2^n x' - E(2^n x')}} + \sum_{n=m}^{\infty} \frac{1}{2^{n-1}}.$$

For when $n < m$

$$u_n = \frac{1}{\sqrt{2^n x' - E(2^n x')}},$$

and when $n \geqq m$,

$$u_n \leqq 2.$$

Therefore the derivative left of $x'$ is always finite. Now since the derivative right of $x'$ is infinite and the derivative left of the point $x'$ is finite, we can say that $f(x)$ has no derivative at the points of the form $x' = \frac{m'}{2^m}$.

## §4. Continuous functions having no derivative at any point of an interval.

Example 1: Weierstrass's problem. Given the function

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n x \pi),$$

in which $x$ is real, $a$ an odd integer, $0 < b < 1$. Then $f(x)$ is continuous and has nowhere a derivative if $ab > 1 + \frac{3\pi}{2}$.

Let $x_0$ be a definite value of $x$, and $m$ an arbitrarily chosen positive integer. Then there will always be a determinate integer $a_m$, for which

$$-\tfrac{1}{2} < a^m x_0 - a_m \leqq \tfrac{1}{2}.$$

We shall now put

$$x_{m+1} = a^m x_0 - a_m,$$

$$x' = \frac{(a_m - 1)}{a^m},$$

$$x'' = \frac{a_m+1}{a^m}.$$

Then $$x' - x_0 = -\frac{(1+x_{m+1})}{a^m}$$

$$x'' - x_0 = \frac{(1-x_{m+1})}{a^m}$$

$$x' < x_0 < x''.$$

This follows from the fact that
$$-\tfrac{1}{2} < x_{m+1} \leqq \tfrac{1}{2}.$$
The integer $m$ can be chosen large enough to insure that $x'$ and $x''$ shall differ from $x_0$ by as small a quantity as we choose. We have then

$$\frac{f(x')-f(x_0)}{x'-x_0} = \sum_{n=0}^{\infty}\left\{ b^n \frac{\cos(a^n x'\pi) - \cos(a^n x_0 \pi)}{x'-x_0}\right\}$$

$$= \sum_{n=0}^{m-1}\left\{(ab)^n \frac{\cos(a^n x'\pi) - \cos(a^n x_0 \pi)}{a^n(x'-x_0)}\right.$$

$$\left. + \sum_{n=m}^{\infty}\right\{ b^n \frac{\cos(a^n x'\pi) - \cos(a^n x_0 \pi)}{x'-x_0}.$$

Putting $n = m+n$ in last part we have

$$= \sum_{n=0}^{m-1} (ab)^n \frac{\cos(a^n x'\pi) - \cos(a^n x_0 \pi)}{a^n(x'-x_0)}$$

$$+ \sum_{n=0}^{\infty} b^{m+n} \frac{\cos(a^{m+n} x'\pi) - \cos(a^{m+n} x_0 \pi)}{x'-x_0}.$$

We know

$$\cos y - \cos x = 2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y).$$

and therefore may write

$$\frac{\cos(a^n x' \pi) - \cos(a^n x_0 \pi)}{a^n(x'-x_0)} = -\pi \sin\left(a^n \frac{x'+x_0}{2}\pi\right) \cdot \frac{\sin\left(a^n \frac{x'-x_0}{2}\pi\right)}{a^n \frac{x'-x_0}{2}\pi}.$$

We know

$$\underset{\Delta x \doteq 0}{L} \frac{\sin \Delta x}{\Delta x} = 1$$

and from this may write

$$-1 \leqq \frac{\sin\left(a^n \frac{x'-x_0}{2}\pi\right)}{a^n \frac{x'-x_0}{2}\pi} \leqq 1.$$

$$\therefore \sum_{n=0}^{m-1} \left\{(ab)^n \frac{\cos(a^n x' \pi) - \cos(a^n x_0 \pi)}{a^n(x'-x_0)}\right\} < \pi \sum_{n=0}^{m-1} (ab)^n$$

$$< \pi \frac{(ab)^m - 1}{ab-1},$$

and if $ab > 1$

$$< \pi \frac{(ab)^m}{ab-1}.$$

Since $a$ is an odd number

$\cos(a^{m+n} x' \pi) = \cos(a^n(\alpha_m - 1)\pi = (-1)^{\alpha_m}$.

To get this last equation we substituted the value of $x'$ and considered $a^n(\alpha_m - 1) = a^n\alpha_m - a^n$.
Since $\sin(a\pi + B) = (-1)^a \sin B$, by substituting the value of $x_0$ we have

$\cos(a^{m+n} x_0 \pi) = \cos(a^n \alpha_m \pi + a^n x_{m+1} \pi) = (-1)^{\alpha_m} \cos(a^n x_{m+1} \pi)$

$$\therefore \sum_0^\infty b^{m+n} \frac{\cos(a^{m+n} x' \pi) - \cos(a^{m+n} x_0 \pi)}{x' - x_0}$$

$$= (-1)^{\alpha_m} (ab)^m \sum_0^\infty \frac{1 + \cos(a^n x_{m+1} \pi)}{1 + x_{m+1}} b^n.$$

all the terms of the summation $\sum_{u=0}^{\infty} b^{u}\left(\frac{1+\cos(a^{u}x_{m+1}\pi)}{1+x_{m+1}}\right)$

are positive, and since $-\frac{1}{2} \leqq x_{m+1} \leqq \frac{1}{2}$, the first term is less than than $2/3$, for $\cos(x_{m+1}\pi)$ cannot be negative.

$$\therefore \frac{f(x')-f(x_0)}{x'-x_0} = (-1)^{a_m}(ab)^m \frac{\eta(2/3 + \pi\eta_1)}{ab-1},$$

where $0 \leqq \eta \leqq 1$, and $-1 \leqq \eta_1 \leqq 1$.

In a similar manner we have

$$\frac{f(x'')-f(x_0)}{x''-x_0} = -(-1)^{a_m}(ab)^m \frac{\eta'(2/3+\pi\eta_1')}{ab-1},$$

where $0 \leqq \eta' \leqq 1$, and $-1 \leqq \eta_1' \leqq 1$. If a and b are so chosen as to make

$$ab > 1 + \frac{3\pi}{2},$$

that is

$$2/3 > \frac{\pi}{ab-1},$$

the two expressions $\frac{f(x')-f(x_0)}{x'-x_0}$ and

$\frac{f(x'')-f(x_0)}{x''-x_0}$ have always opposite signs,

and are both infinitely great when m increases indefinitely. Hence $f(x)$ possesses neither a definite finite or infinite derivative.

For a discussion of problems of this type where definite values

are given a and b see page 124 of Dini's Functionen Theorie.

For the developement of a general problem which will have at no point a derivative see Dini's Functionen Theorie pages 206 - 13.

# Chapter IV

## Geometrical Representation of Functions Which are Continuous but have no Derivative.

### §1. Steinitz' method of constructing a continuous function which does not have a derivative at any point.

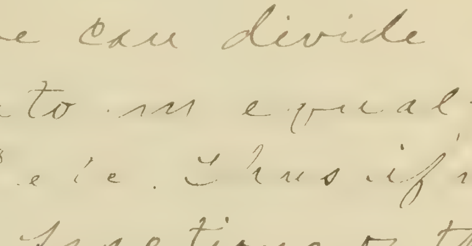
(a). To construct a function defined for every point of an interval. Let the region $A$ include all the numbers between $p$ and $q$, inclusive of the limits, and let $f(x)$ be a continuous function defined for this interval. If $a_0, a_1, a_2$ --- is a sequence of numbers in $A$, whose limit is $a$, then from our definition of continuity on page 3, we know that $f(a_0), f(a_1), f(a_2)$ --- form a series whose limit is $f(a)$. Let $B$ be a region contained in $A$, such that every number in $A$ can be represented as the limit of a sequence of numbers of $B$. If $A$ includes all the numbers from 0 to 1, and $B$ includes all rational

numbers from 0 to 1, then B is a region contained in the region A, such that every number of A can be represented as the limit of a sequence of numbers in B. Now from our definition of continuity, we see that the continuous function $f(x)$ for the values which it receives in the region B, must agree with the values which it receives in the entire region A. If $\varphi(x)$ is a continuous function, defined for the region B, i.e. every sequence of numbers $b_0, b_1, b_2, \ldots$ in B, has from the definitions of our regions, a limit a in the region A, and since $\varphi(x)$ is defined and continuous for every point of the region B, there is corresponding to this sequence, a sequence of numbers $\varphi(b_0), \varphi(b_1), \varphi(b_2)$ whose limit is $f(a)$. Therefore since this limit must always be the same, we can say that there is one and only one function $f(x)$, defined and continuous in the region A, and agreeing with $\varphi(x)$, defined and continuous in the region B.

Let the region A include all the

numbers between 0 and 1, and let the region $B$ include all the rational fractions of the form $\frac{1}{m^k}$. Now since $B$ is contained in $A$, and every number in $A$ can be represented as the limit of a sequence of numbers in $B$, to obtain a function $f(x)$ continuous and defined in the region $A$, we have only to form a function $\varphi(x)$, continuous and defined in the region $B$.

I shall confine myself to functions which vanish at the point $x=0$, for all other functions can be derived from these through the addition of a constant. Let $\varphi(x)$ be a function defined for the rational fractions of the form $\frac{1}{m^k}$, in the region $B$ between 0 and 1. We can arrange the numbers of $B$ so we can divide the region 0---1 first into $m$ equal parts, then into $m^2$, then $m^3$ etc. Thus if $m=2$ we have the rational fractions of the form $\frac{1}{2^k}$. If $k=0$, the division gives the numbers 0 and 1: If $k=1$ the division gives the numbers $0, \frac{1}{2}, 1$. If $k=2$



the division gives the points $0, \frac{1}{4}, \frac{1}{2}, 1$. We would thus give $k$ all the values from $0$ to $\infty$. For the general case we would have the points as follows:

$0 \quad - \quad - \quad - \quad - \quad - \quad 1$

$0 \quad \frac{1}{m} \quad \frac{2}{m} \quad - \quad - \quad \frac{m-1}{m}, \; 1$

$0 \quad \frac{1}{m^2} \quad \frac{2}{m^2}, \; - - \; \frac{1}{m}, \; \frac{m+1}{m^2} \; - - \; \frac{2}{m}, \; - - \; \frac{m^2-1}{m^2}, \; 1.$

$- \; - \; - \; - \; - \; - \; -$

If $k$ is given all the values from $0$ to $\infty$ we should get all the rational fractions of the form $\frac{l}{m^k}$. (See Stolz allgemeine Arithmetik p 17.)

Let $\varphi\left(\frac{l}{m^k}\right)$ be the value of $\varphi(x)$ at the point $x = \frac{l}{m^k}$. Now put,

$$1. \begin{cases} \varphi(1) - \varphi(0) = \Delta_{0,0} \\ \varphi\left(\frac{1}{m}\right) - \varphi(0) = \Delta_{1,0} \\ \varphi\left(\frac{2}{m}\right) - \varphi\left(\frac{1}{m}\right) = \Delta_{1,1} \\ \varphi(1) - \varphi\left(\frac{m-1}{m}\right) = \Delta_{1,m-1} \end{cases}$$

In general

$$\varphi\left(\frac{l+1}{m^k}\right) - \varphi\left(\frac{l}{m^k}\right) = \Delta_{k,l} \quad \left(\begin{matrix} k = 0, 1, \\ l = 0, 1, - - (m^k - 1) \end{matrix}\right)$$

Now since we know $\varphi(0) = 0$, by hypothesis, if the differences $\Delta_{k,l}$ are given we can at once determine the values of the function from the above equation. If we add all except the first of equations 1, we get

$$\varphi(1) - \varphi(0) = \Delta_{1,0} + \Delta_{1,1} + - - - + \Delta_{1,m-1}.$$

From this and the first of equations 1 we

have

(2) $\Delta_{0,0} = \Delta_{1,0} + \Delta_{1,1} + \cdots + \Delta_{1,m-1}$.

This would define our function for only $m$ points. If now we divide each of $\Delta_{1,0}, \Delta_{1,1} \ldots \Delta_{m-1}$ into $m$ parts, we should have our function defined for $m^2$ points and we would get in a similar manner to equation (2)

$$(3)\left\{\begin{array}{l} \Delta_{1,0} = \Delta_{2,0} + \Delta_{2,1} + \cdots + \Delta_{2,m-1} \\ \text{-- -- -- -- -- -- -- -- -- --} \\ \Delta_{1,m-1} = \Delta_{2,2m-2} + \Delta_{2,2m-1} + \cdots + \Delta_{2,3m-1}. \end{array}\right.$$

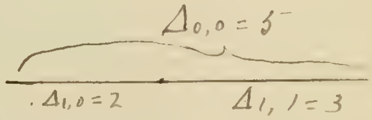
If we continued this division of the $\Delta$'s we should finally get a function defined for every rational fraction of the form $\frac{1}{m^k}$, and would have a relation between the $\Delta$'s analogous to that of equations (2) and (3).

$$\Delta_{k,l} = \Delta_{k+1,lm} + \Delta_{k+1,lm+1} + \cdots + \Delta_{k+1,lm+m-1}.$$

We know nothing, however, of the continuity of this function but simply know it is defined for each point of the interval $B$. I shall take a special example to illustrate this.

Let $m = 2$, $\Delta_{1,0} = 2$, $\Delta_{1,1} = 3$.

Then from equation (2)

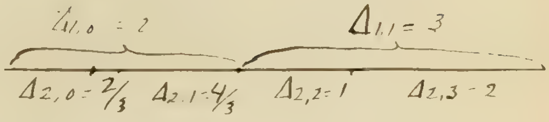



$\Delta_{0,0} = 2 + 3 = 5$.

Making $k = 2$ and dividing our $\Delta$'s we have

$\Delta_{2,0} = \frac{2}{3}$ $\Delta_{2,1} = \frac{4}{3}$, $\Delta_{2,2} = 1$, $\Delta_{2,3} = 2$.

These divisions are purely arbitrary as long as they fulfill the conditions




$$\Delta_{1,0} = \Delta_{2,0} + \Delta_{2,1} = 2$$

$$\Delta_{1,1} = \Delta_{2,2} + \Delta_{2,3} = 3.$$

Making $k = 3$, we can choose our divisions in any way so the following conditions are fulfilled

$$\Delta_{2,0} = \Delta_{3,0} + \Delta_{3,1} = \frac{2}{3}$$

$$\Delta_{2,1} = \Delta_{3,2} + \Delta_{3,3} = \frac{4}{3}$$

$$\Delta_{2,2} = \Delta_{3,4} + \Delta_{3,5} = 1$$

$$\Delta_{2,3} = \Delta_{3,6} + \Delta_{3,7} = 2.$$

Therefore, we may put

$\Delta_{3,0} = \frac{1}{8}$, $\Delta_{3,1} = \frac{13}{24}$, $\Delta_{3,2} = \frac{1}{3}$, $\Delta_{3,3} = 1$, $\Delta_{3,4} = \frac{1}{4}$, $\Delta_{3,5} = \frac{3}{4}$, $\Delta_{3,6} = \frac{2}{3}$, $\Delta_{3,7} = \frac{4}{3}$

Then from equations 1 we have for $k = 0$

$$\varphi(1) - \varphi(0) = \Delta_{0,0}$$

$$\varphi(1) = 0 + 5 = 5.$$

For $k = 1$ we have

$$\varphi(\tfrac{1}{2}) - \varphi(0) = \Delta_{1,0}$$

$$\varphi(\tfrac{1}{2}) = 0 + 2 = 2$$

$$\varphi(1) - \varphi(\tfrac{1}{2}) = \Delta_{1,1}$$

$$\varphi(1) = 2 + 3 = 5.$$

For $k = 2$, we have

$\varphi(\frac{1}{4}) - \varphi(0) = \Delta_{2,0}$

$\varphi(\frac{1}{4}) = 0 + \frac{2}{3} = \frac{2}{3}$

$\varphi(\frac{1}{2}) - \varphi(\frac{1}{4}) = \Delta_{2,1}$

$\varphi(\frac{1}{2}) = \frac{2}{3} + \frac{4}{3} = 2$

$\varphi(\frac{3}{4}) - \varphi(\frac{1}{2}) = \Delta_{2,2}$

$\varphi(\frac{3}{4}) = 2 + 1 = 3$

$\varphi(1) - \varphi(\frac{3}{4}) = \Delta_{2,3}$

$\varphi(1) = 5.$

Similarly for $k = 3$ we should get
$\varphi(\frac{1}{8}) = \frac{1}{8}$, $\varphi(\frac{1}{4}) = \frac{13}{24}$, $\varphi(\frac{3}{8}) = 1$, $\varphi(\frac{1}{2}) = 2$, $\varphi(\frac{5}{8}) = \frac{8}{4}$,
$\varphi(\frac{3}{4}) = 3$, $\varphi(\frac{7}{8}) = 3\frac{2}{3}$, $\varphi(1) = 5.$

See figure 1 on following page for the different approximation curves.

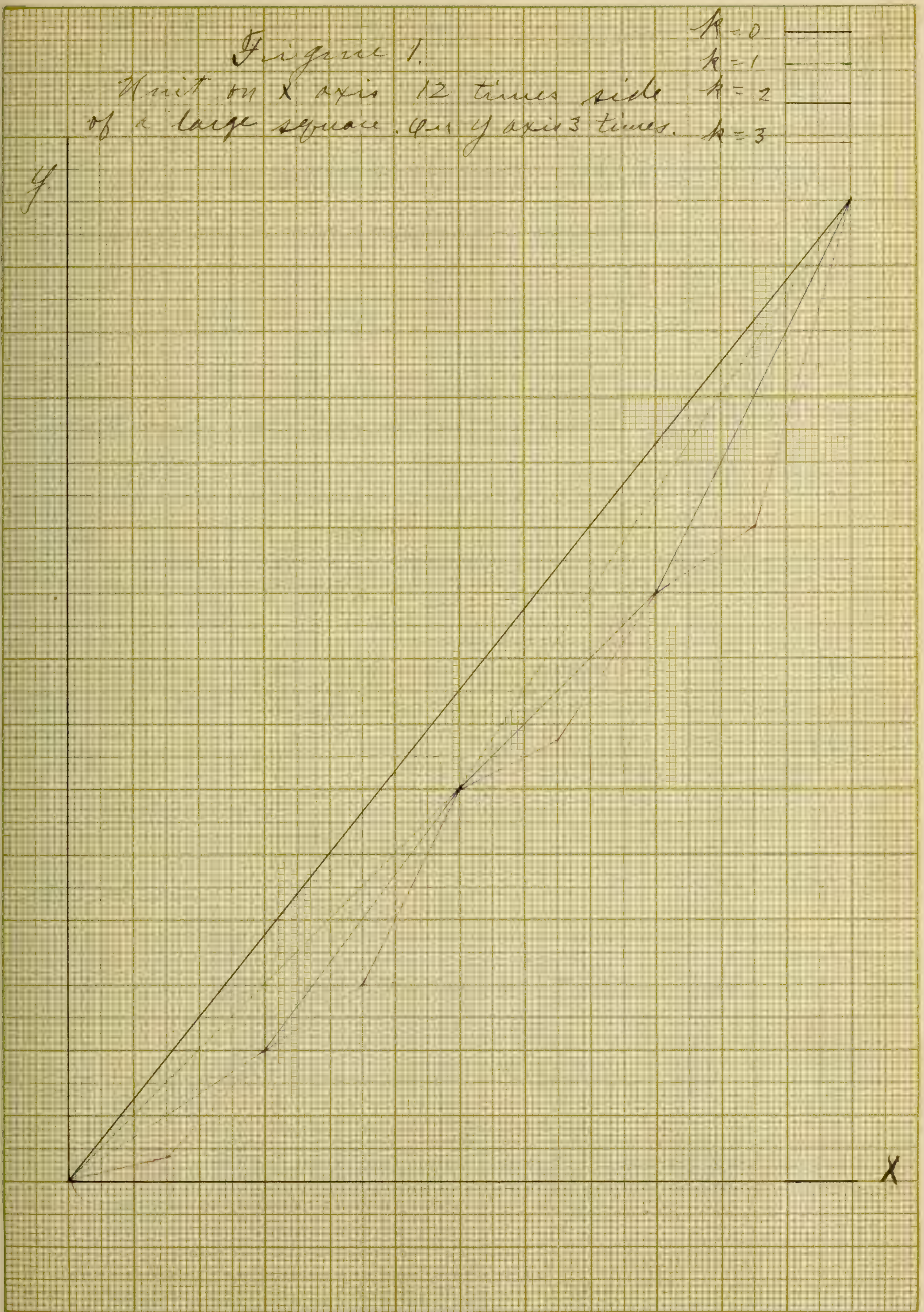


Figure 1.
Unit on x axis 12 times side
of a large square. On y axis 3 times.
k = 0
k = 1
k = 2
k = 3
y.
X

(b) To construct $\varphi(x)$, which is now defined for the region $R$, so that it will be continuous in the region. Let the greatest of the absolute values of $\Delta_{k,0}, \Delta_{k,1}, \cdots \Delta_{k,m^k-1}$ be designated by $\alpha_k$. Since each $\Delta$ is the difference between two values of $\varphi(x)$, by the definition of $\Delta$, $\alpha_k$ is an increment of $\varphi(x)$ corresponding to the increment $\frac{1}{m^k}$ in the variable, and since

$$\lim_{k \doteq \infty} \frac{1}{m^k} = 0,$$

a necessary condition for the continuity of $\varphi(x)$ is that

$$\lim_{k \doteq \infty} \alpha_k = 0.$$

This follows directly from our definition of continuity on page 3. That this is not a sufficient condition, however, will come out in the proof of the following theorem.

If the sequence of values $\alpha_0, \alpha_1, \alpha_2$ which we shall call the maximal differences, not only satisfies the condition

(1) $$\lim_{k \doteq \infty} \alpha_k = 0,$$

but also the condition that their sum

(2) $\alpha_0 + \alpha_1 + \alpha_2 + \cdots\cdots$

converges then the function $\varphi(x)$ defined in $\mathfrak{B}_1(a)$, is continuous.

Since the series (2) is by hypothesis convergent, if $\mu_i$ represents the sum of all the terms after the $i$ th. term, we know

(3). $$\lim_{i=\infty} \mu_i = 0.$$

Let $\frac{p}{m^k}$ and $\frac{q}{m^k}$ be any two numbers in the region $\mathfrak{B}$, then, since $\alpha_k$ is the greatest increment in $\varphi(x)$ for an increment of $\frac{1}{m^k}$ in the variable, we can write

(4) $$|\varphi(\tfrac{q}{m^k}) - \varphi(\tfrac{p}{m^k})| \leqq (p-q)\,\alpha_k$$

(5) Let $$\frac{l}{m^k} \leqq x \leqq \frac{l+1}{m^k}.$$

If $x < \frac{l+1}{m^k}$, $x$ can be expressed in the form:

(6) $$x = \frac{l}{m^k} + \frac{c_1}{m^{k+1}} + \cdots + \frac{c_r}{m^{k+r}}.$$

(See Stolz Allgemeine Arithmetic p. 77)

If $x > \frac{l}{m^k}$, $x$ can be expressed in the form.

(7) $$x = \frac{l+1}{m^k} - \left(\frac{c'_1}{m^{k+1}} + \cdots + \frac{c'_r}{m^{k+r}}\right).$$

Where the coefficients $c$ and $c'$ are integral numbers satisfying the

conditions

(8) $$0 \leq c \leq m-1$$
$$0 \leq c' \leq m-1.$$

(9) Let $\frac{l}{m^k} = x_0$, $\frac{l+1}{m^k} = x'_0$,

and

(10) $$\frac{l}{m^k} + \frac{c_1}{m^{k+1}} + \cdots + \frac{c_h}{m^{k+h}} = x_h \quad (h = 0 \cdots n),$$

and

(11). $$\frac{l+1}{m^k} - \left(\frac{c'_1}{m^{k+1}} + \cdots + \frac{c'_h}{m^{k+h}}\right) = x'_h.$$

Putting equation (10) over a common denominator we have

(12) $$x_h = \frac{l\,m^h + c_1 m^{h-1} + \cdots + c_{h-1}\,m + c_h}{m^{k+h}}.$$

If now we write out $x_{h-1}$ analogous to $x_h$ and multiply both numerator and denominator by $m$, we have

(13) $$x_{h-1} = l m^h + c_1 m^{h-1} + \cdots + c_{h-1}\,m.$$

In a similar manner we get for the $x'$s from equation (11)

(14) $$x'_h = \frac{(l+1)m^h - (c'_1\, m^{h-1} + \cdots + c'_{h-1}\, m) - c'_h}{m^{k+h}},$$

and

(15) $$x'_{h-1} = \frac{(l+1)m^h - (c'_1\, m^{h-1} + \cdots + c'_{h-1}\, m)}{m^{k+h}}.$$

Subtracting (12) from (11) we get

(16) $$x_h - x_{h-1} = \frac{c_h}{m^{k+h}}.$$

Subtracting (15) from (14) we have

(17) $\quad x'_h - x'_{h-1} = \frac{c'_h}{m^{k+h}}$.

Now, since by definition $\alpha_{k+h}$ is the maximum increment of $\varphi(x)$ corresponding to an increment $\frac{1}{m^{k+h}}$ in the variable, we can write

(18) $|\varphi(x_h) - \varphi(x_{h-1})| \leqq c_h \alpha_{k+h} \leqq (m-1)\alpha_{k+h}$

$|\varphi(x'_h) - \varphi(x'_{h-1})| \leqq c'_h \alpha_{k+h} \leqq (m-1)\alpha_{k+h}$;

for by definition $c$ and $c'$ are less or equal to $(m-1)$. If now we give to $h$ all values from $0$ to $r$ in equations (18) and take the sum of all these, we have, by replacing the sum of the terms

$\alpha_k + \alpha_{k+1} + \quad\cdots\quad + \alpha_{k+r}$

by $u_k$, which by definition was the sum of all the terms after the $k$th,

(19) $|\varphi(x_r) - \varphi(x_0)| \leqq (m-1)u_k$

$|\varphi(x'_r) - \varphi(x'_0)| \leqq (m-1)u_k$.

Since $x_r$ and $x'_r$ may be any values in the interval to which $x$ is confined, we can replace them by $x$, and giving $x_0$ and $x'_0$ their values from equation (9), we have

(20) $|\varphi(x) - \varphi(\frac{l}{m^k})| \leqq (m-1)u_k$

$|\varphi(x) - \varphi(\frac{l+1}{m^k})| \leqq (m-1)u_k$.

Let $x_1$ and $x_2$ be two numbers in $\mathcal{B}$, such that $x_2 \geqq x_1$; and $x_2 - x_1 < \frac{1}{m^k}$. Then we can always choose $l$ arbitrarily so that

$$\frac{l}{m^k} \leqq x_1 \leqq \frac{l+1}{m^k},$$

and since $x_2 - x_1 < \frac{1}{m^k}$, we know either

$$\frac{l}{m^k} \leqq x_2 \leqq \frac{l+1}{m^k},$$

or

$$\frac{l+1}{m^k} \leqq x_2 \leqq \frac{l+2}{m^k}.$$

Then we can substitute $x_2$ and $x_1$ for $x$ in equations (20) and we have

$$(21) \quad \left|\varphi\left(\tfrac{l+1}{m^k}\right) - \varphi(x_1)\right| \leqq (m-1)\, M_k,$$

$$\left|\varphi(x_2) - \varphi\left(\tfrac{l+1}{m^k}\right)\right| \leqq (m-1)\, M_k.$$

adding the last two equations we have

$$(22) \quad |\varphi(x_2) - \varphi(x_1)| \leqq 2(m-1)\, M_k.$$

But according to equation (3)

$$\lim_{k=\infty} 2(m-1)\, M_k = 0,$$

and the continuity of $\varphi(x)$ is proven. That the convergence of the sum

$$\alpha_0 + \alpha_1 + \alpha_2 + \cdots$$

is a necessary condition for continuity is evident, for otherwise the right hand member of equation (22) would not approach zero as $k \doteq \infty$.

To get a representation of a continuous function, we shall take $\Delta_{0,0} \neq 0$

and divide it arbitrarily into $m$ parts $\Delta_{1,0}, \Delta_{1,1}, \quad \Delta_{1,m-1}$, each part different from zero. Divide each of these $\Delta$'s into $m$ parts which bear the same relation to it as the $m$ parts of the first division and continue this process indefinitely. We shall then have the relations

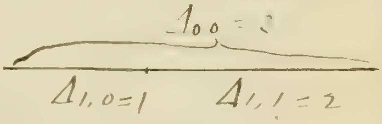
$$\Delta_{1,0} + \cdots + \Delta_{1,m-1} = \Delta_{0,0}$$

$$\Delta_{k+1,lm} + \cdots + \Delta_{k+1,lm+m-1} = \Delta_{k,l}.$$

$$\Delta_{k+1,lm} : \Delta_{k+1,lm+1} : \cdots : \Delta_{k+1,lm+m-1} :: \Delta_{1,0} : \Delta_{1,1} : \cdots : \Delta_{1,m-1}$$

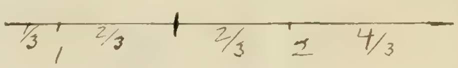
$$\Delta_{k+1,lm} = \frac{\Delta_{k,l}}{\Delta_{0,0}} \Delta_{1,0}.$$

For example, let $\Delta_{0,0} = 3$ and divide it into the two parts $\Delta_{1,0} = 1$, $\Delta_{1,1} = 2$. We must now divide each of these into 2 parts proportional to those of the first division. then let




$$\Delta_{1,0} = \Delta_{2,0} + \Delta_{2,1} = \tfrac{1}{3} + \tfrac{2}{3} = 1$$

$$\Delta_{1,1} = \Delta_{2,2} + \Delta_{2,3} = \tfrac{2}{3} + \tfrac{4}{3} = 2$$




and continue this division indefinitely. This is called the periodic method of division.

The periodic method of division will give a continuous function if $|\Delta_{0,0}| > |\Delta_{1,l}|$ (any $\Delta$ of the first division): The maximal dif-

-ferences $\alpha_0 - - - - \alpha_k$, by this meth-od of division form a geometrical ratio, and the conditions for continuity, viz:

$$\underset{k=\infty}{L} \alpha_k = 0$$

and the convergence of the sum will be fulfilled, if this geometrical series is a decreasing series. It will always be a decreasing series if $\alpha_0 > \alpha_1$; that is, if the first term is greater than the second. The first term of the series will always be greater than the second, if $|\Delta_{0,0}|$ is greater than the absolute value of any $\Delta$ of the first division; for these have to be divid-ed in a similar manner and thus could not yield any division greater than the maximum of the first divis-ion.

If we let $\Delta_{1,0} = \delta_1$, $\Delta_{1,1} = \delta_2$, $- - - \Delta_{1,m-1} = \delta_m$, then $\delta_1, \delta_2 - - - \delta_m$ will determine a function, defined and continuous in the region $\mathcal{B}$, which we shall call $\varphi(x; \delta_1 - - - \delta_m)$. Now corresponding to this, we know from §1,(1), that we shall have a function $f(x; \delta_1 - - \delta_m)$ defined and continuous in the region $\mathcal{A}$ and agreeing with $\varphi(x; \delta_1 - - \delta_m)$ at every point of the region.

Example: Given the function $f(x; 1, 2)$. This function is continuous; for

$$|1+2| = 3,$$

which is greater than any $\delta$ the condition for continuity by the theorem just proven. Getting the divisions as in the example on page 75, and then the values of the function from these as in ~~the~~ the example on page 67, we have:

| $k=0$ | | $k=1$ | | $k=2$ | | $k=3$ | |
|---|---|---|---|---|---|---|---|
| $x$ | $f(x)$ | $x$ | $f(x)$ | $x$ | $f(x)$ | $x$ | $f(x)$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | | | | | 1/8 | 4 |
| | | | | 1/4 | 1/3 | 2/8 | 1/3 |
| | | | | | | 3/8 | 5/4 |
| | | 1/2 | 1 | 2/4 | 1 | 4/8 | 1 |
| | | | | | | 5/8 | 11/4 |
| | | | | 3/4 | 5/3 | 6/8 | 5/3 |
| | | | | | | 7/8 | 19/4 |
| 1 | 3 | 1 | 3 | 1 | 3 | 1 | 3 |

For the different approximation curves see figure 2 on following page.

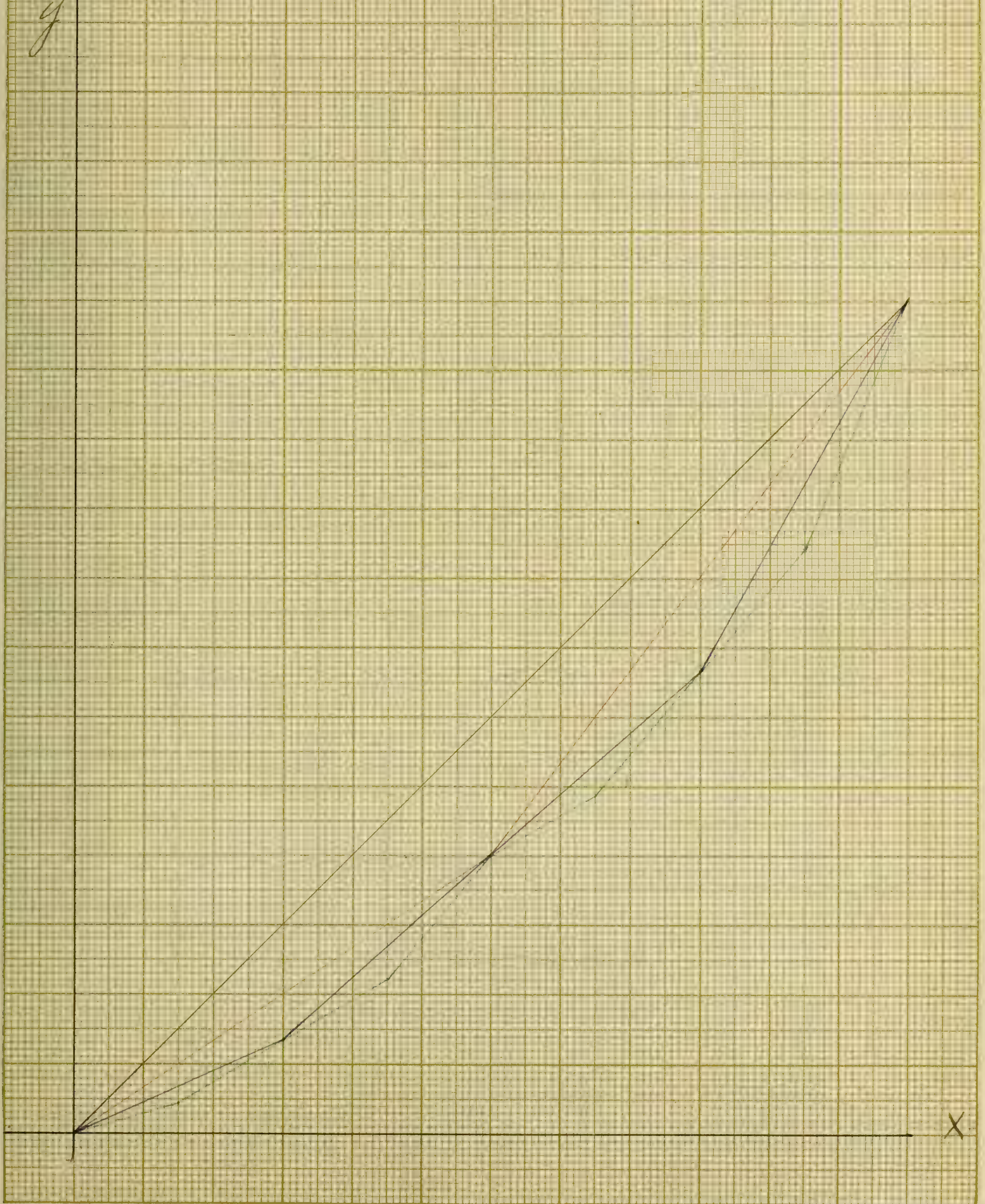
Figure 2.

f(x; 1, 2)

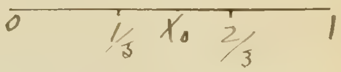
Unit on x axis 12 ——

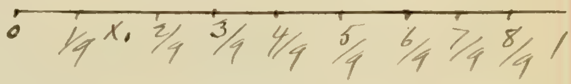
Unit on y axis 4 ——

k=0 ——
k=1 ——
k=2 ——
k=3 ——



(C) To construct a continuous function which has at no point a finite derivative. We can decrease the interval inclosing the point $x_0$ by allowing $k$ to so vary that it passes through all the positive integers, and finding the integral number $l_k$ so that $x_0$ lies between $\frac{l_k}{m^k}$ and $\frac{l_k+1}{m^k}$, including the lower limit and, if $x_0 \neq 1$, excluding the upper limit. Thus suppose that $m = 3$, and $l_1 = 1$, then $x_0$ lies between $1/3$ and $2/3$. Now, when $k = 2$, if we suppose $l_2 = 1$, $x_0$ will lie between $\frac{1}{9}$ and $2/9$, and as we increase $k$ indefinitely the interval inclosing $x_0$ will decrease indefinitely.




If $\varphi(x)$ and $f(x)$ are defined as in §1 (a) pages 65-6, and if

$$x_1 = \frac{l_k}{m^k}, \quad x_2 = \frac{l_k+1}{m^k}$$

we have

(1) $$x_2 - x_1 = \frac{1}{m^k}.$$

Then from our definition of $f(x)$, we can put

$$(2) \qquad f(x_2) - f(x_1) = \Delta k, l_k.$$

Dividing equation (2) by (1) we have

$$(3) \qquad \frac{f(x_2) - f(x_1)}{x_2 - x_1} = m^k \Delta k, l_k.$$

Therefore, we see at once that there will exist no finite derivative, if

$$(4) \qquad \lim_{k \doteq \infty} m^k (\Delta k, l_k) = \infty.$$

If $\beta_k$ represents the lower limit of the absolute values of the differences taken, then

$$(5) \qquad \frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq m^k \beta_k.$$

Therefore it follows directly from (4), that if

$$(6) \qquad \lim_{k \doteq \infty} m^k \beta_k = \infty,$$

$f(x)$ will have at no point a finite derivative.

In order to fullfill the condition of continuity, we know that the sum of the maximal differences

$$(7) \quad \alpha_0 + \alpha_1 + \alpha_2 + \cdots\cdots,$$

must form a convergent series. Then the lower limits of the differences

(8) $\beta_0 + \beta_1 + \beta_2 + - - -$,
will certainly give a convergent sum. If now in addition to this, the terms $\beta_0, m\beta_1, m^2\beta_2 - - - m^k\beta_k$ increase indefinitely, we know from (6) that $f(x)$ will not have a finite derivative at any point. Now the terms of the series just given form a geometrical series, and therefore will become infinite if the second term is greater than the first; that is, if

$$m\beta_1 > \beta_0.$$

From our definition of $\beta$ we know $\beta_0 = |\Delta_{0,0}|$, and $\beta_1$ is the least of the $|\Delta|$ of the first division. Therefore we can say that a periodic will give a continuous function which does not have a finite derivative, if each of the $m$ parts, into which $\Delta_{0,0}$ is divided, is less than the absolute value of $\Delta_{0,0}$, but greater than the absolute value of $\frac{1}{m}\Delta_{0,0}$.

Example: Given the function $f(x; 1, 1, 1, -1)$. This function is continuous but has at no point a derivative; for it

-ful fills the conditions just given above, since

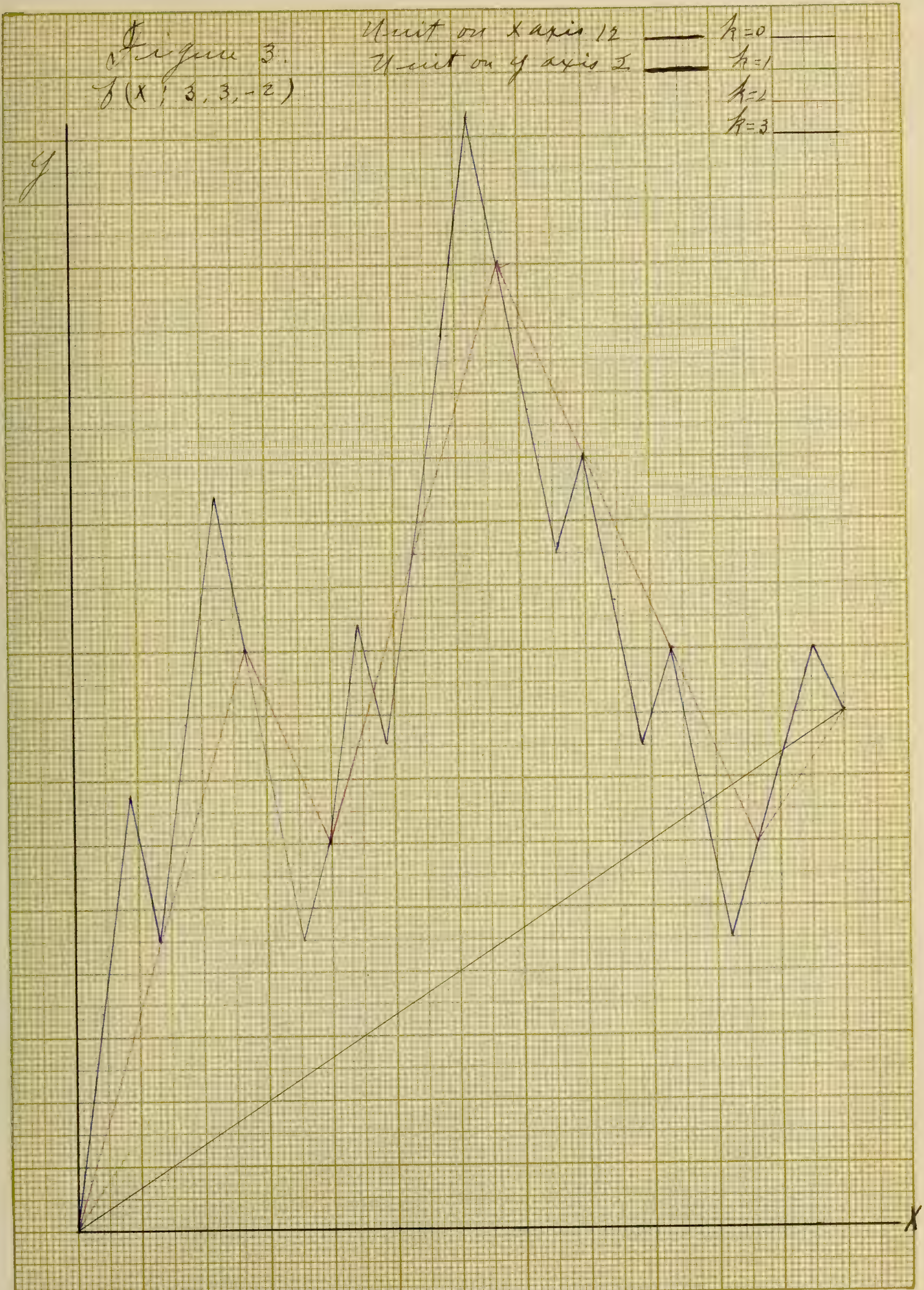
$$|\delta_1+\delta_2+\delta_3+\delta_4| = |1+1+1-1| = 2 > |\delta_i|$$

$$\frac{1}{m}|\delta_1+\delta_2+\delta_3+\delta_4| = \frac{1}{4}|1+1+1-1| = \frac{1}{2} < |\delta_i|.$$

Finding the values of the function as in previous problems, we have:

| | $k=0$ | | $k=1$ | | $k=2$ | | |
|---|---|---|---|---|---|---|---|
| | $x$ | $f(x)$ | $x$ | $f(x)$ | $x$ | $f(x)$ | |
| | 0 | 0 | 0 | 0 | 0 | 0 | |
| | | | | | 1/16 | 1/2 | |
| | | | | | 2/16 | 1 | |
| | | | | | 3/16 | 3/2 | |
| | | | 1/4 | 1 | 4/16 | 1 | |
| | | | | | 5/16 | 3/2 | |
| | | | | | 6/16 | 2 | |
| | | | | | 7/16 | 5/2 | |
| | | | 2/4 | 2 | 8/16 | 2 | |
| | | | | | 9/16 | 5/2 | |
| | | | | | 10/16 | 3 | |
| | | | | | 11/16 | 7/2 | |
| | | | 3/4 | 3 | 12/16 | 3 | |
| | | | | | 13/16 | 5/2 | |
| | | | | | 14/16 | 2 | |
| | | | | | 15/16 | 3/2 | |
| | 1 | 2 | 1 | 2 | [illegible] | 2 | |

See figure 3 on following page.

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Example: Given the function $f(x; 3, 3, -2)$. This function is continuous and does not have at any point a derivative; for

$$|\delta_1 + \delta_2 + \delta_3| = |3 + 3 - 2| = 4 > |\delta_i|,$$

$$\frac{1}{m}|\delta_1 + \delta_2 + \delta_3| = \frac{1}{3}|3 + 3 - 2| = 4/3 < |\delta_i|.$$

Finding the values of the function as in previous examples, we have.

| $k=0$ | | $k=1$ | | $k=2$ | | $k=3$ | |
|---|---|---|---|---|---|---|---|
| $x$ | $f(x)$ | $x$ | $f(x)$ | $x$ | $f(x)$ | $x$ | $f(x)$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | | | | | $1/27$ | $27/16$ |
| | | | | | | $2/27$ | $27/8$ |
| | | | | $1/9$ | $18/8$ | $3/27$ | $18/8$ |
| | | | | | | $4/27$ | $63/16$ |
| | | | | | | $5/27$ | $90/16$ |
| | | | | $2/9$ | $18/4$ | $6/27$ | $18/4$ |
| | | | | | | $7/27$ | $27/8$ |
| | | | | | | $8/27$ | $9/4$ |
| | | $1/3$ | 3 | $3/9$ | 3 | $9/27$ | 3 |
| | | | | | | $10/27$ | $75/16$ |
| | | | | | | $11/27$ | $102/27$ |
| | | | | $4/9$ | $21/4$ | $12/27$ | $21/4$ |
| | | | | | | $13/27$ | $111/16$ |
| | | | | | | $14/27$ | $138/16$ |

| $k=0$ | | $k=1$ | | $k=2$ | | $k=3$ | |
|---|---|---|---|---|---|---|---|
| $x$ | $f(x)$ | $x$ | $f(x)$ | $x$ | $f(x)$ | $x$ | $f(x)$ |
| | | | | $5/9$ | $30/4$ | $15/27$ | $30/4$ |
| | | | | | | $16/27$ | $51/8$ |
| | | | | | | $17/27$ | $42/8$ |
| | | $2/3$ | $6$ | $6/9$ | $6$ | $18/27$ | $6$ |
| | | | | | | $19/27$ | $39/8$ |
| | | | | | | $20/27$ | $30/8$ |
| | | | | $7/9$ | $9/2$ | $21/27$ | $9/2$ |
| | | | | | | $22/27$ | $27/8$ |
| | | | | | | $23/27$ | $18/8$ |
| | | | | $8/9$ | $3$ | $24/27$ | $3$ |
| | | | | | | $25/27$ | $15/4$ |
| | | | | | | $26/27$ | $18/4$ |
| $1$ | $4$ | $1$ | $4$ | $1$ | $4$ | $1$ | $4$ |

For the geometrical representation see figure 4, page

Figure 4

$f(x; 1, 1, 1, -1)$

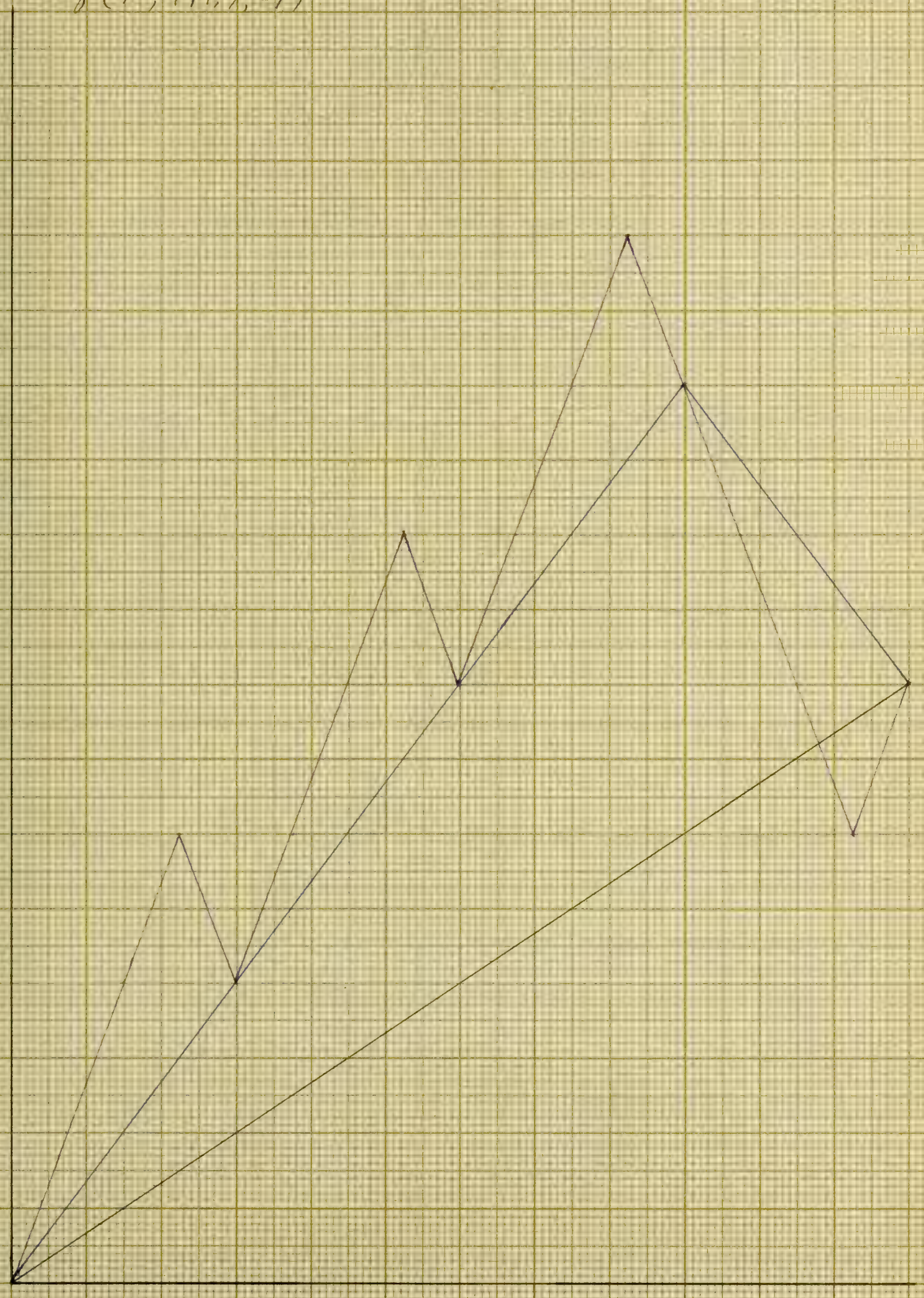
Unit on x axis 12

Unit on y axis 4

$k=0$

$k=1$

$k=2$



(c) To construct a function which is continuous but does not have either a finite or infinite derivative at any point. – The conditions discussed under (b) do not exclude the possibility of an infinite derivative, but only one further condition is necessary to exclude this possibility and this condition can be best explained by an example.

Given the function $f(x; 4, 4, -5, -5, 4, 4)$. This function is continuous and has no finite derivative; for

$|\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6| = |4+4-5-5+4+4| = 6 > |\delta_i|$,

$\frac{1}{6}|\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6| = \frac{1}{6}|4+4-5-5+4+4| = 1 < |\delta_i|$.

In this example $m = 6$, $\Delta_{0,0} = 6$, $\Delta_{1,0} = 4$, $\Delta_{1,1} = 4$, $\Delta_{1,2} = -5$, $\Delta_{1,3} = -5$, $\Delta_{1,4} = 4$, $\Delta_{1,5} = 4$ by definition. We shall now form sums containing 2, 3, – and finally $m$ $\Delta$'s, the terms in each sum being adjacent ones as follows:

$\Delta_{1,0} + \Delta_{1,1}$, $\Delta_{1,1} + \Delta_{1,2}$, – – – – – – $\Delta_{1,m-2} + \Delta_{1,m-1}$.

$\Delta_{1,0} + \Delta_{1,1} + \Delta_{1,2}$, – – – – – – ; $\Delta_{1,m-3} + \Delta_{1,m-2} + \Delta_{1,m-1}$.

– – – – – – – – – –

$\Delta_{1,0} + \Delta_{1,1} + $ – – – – – $+ \Delta_{1,m-1}$.

With each difference $\Delta_{1,l}$ we shall

now associate one of these sums, just given, which contains $\Delta_{1,l}$, and designate this sum by $\Delta'_{1,l}$. In this example, we can determine $\Delta'_{1,l}$ so that $\Delta'_{1,l}$ and $\Delta_{1,l}$ have opposite signs. To illustrate this take

$$
\begin{aligned}
\Delta'_{1,0} &= \Delta_{1,0} + \Delta_{1,1} + \Delta_{1,2} + \Delta_{1,3} &&= -2, \\
\Delta'_{1,1} &= \Delta_{1,1} + \Delta_{1,2} &&= -1, \\
(1) \quad \Delta'_{1,2} &= \Delta_{1,0} + \Delta_{1,1} + \Delta_{1,2} &&= 3, \\
\Delta'_{1,3} &= \Delta_{1,3} + \Delta_{1,4} + \Delta_{1,5} &&= 3, \\
\Delta'_{1,4} &= \Delta_{1,3} + \Delta_{1,4} &&= -1, \\
\Delta'_{1,5} &= \Delta_{1,2} + \Delta_{1,3} + \Delta_{1,4} + \Delta_{1,5} &&= -2.
\end{aligned}
$$

Whenever $\Delta'_{1,l}$ can thus be determined so that its sign is opposite to that of $\Delta_{1,l}$, and the conditions in §1(b) are fulfilled we have a continuous function, which does not have either a finite or infinite derivative; for, $\Delta'_{1,l}$ represents the increment of the function, corresponding to an increment $\frac{\varepsilon_l}{m}$ in the variable, where $\varepsilon_l$ is the number of terms in $\Delta'_{1,l}$. Therefore we may write

$$(2) \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\Delta'_{1,l}}{\frac{\varepsilon_l}{m}} \qquad (l = 0, 1, \cdots m-1).$$

$$= \frac{1}{\varepsilon_l} \cdot m\,\Delta \,.$$

where $x_2 - x_1 = \frac{\varepsilon_l}{m}$.

If now we let

(3) $\qquad \alpha_l = \frac{1}{\varepsilon_l} \cdot \frac{\Delta'_{1,l}}{\Delta_{1,l}}$,

by clearing of fractions and multiplying by $m$ ~~we~~ have

(4) $\qquad \frac{1}{\varepsilon_l} m.\ \Delta'_{1,l} = \alpha_l .\ m\, \Delta_{1,l}$.

Since $\Delta_{1,l}$ and $\Delta'_{1,l}$ are always of opposite signs, $\alpha_0, \alpha_1, --- \alpha_{m-1}$ are all negative numbers. If $\Delta'_{k+1,lm}, \Delta'_{k+1,lm+1}, ---- \Delta'_{k+1,lm+m-1}$, are the values obtained by replacing $\Delta_{1,s}$, $\Delta'_{1,s}$ $(s = 0, 1 \dots m-1)$ by $\Delta_{k+1,lm+s}$ and $\Delta'_{k+1,lm+s}$ in equations (1), we know from equation (1) page 75 that

$$\frac{\Delta'_{k+1,\,lm+s}}{\Delta_{k+1,\,lm+s}} = \frac{\Delta'_{1,s}}{\Delta_{1,s}} .$$

Every value $\Delta_{k,l}$ represents an increment of the function corresponding to $\frac{1}{m^k}$ the increment of the variable. Then the derivative corresponding to $\Delta_{k,l}$ will be $m^k \Delta_{k,l}$. Every value $\Delta'_{k,l}$ represents an increment of the function corresponding to an increment $\frac{\varepsilon_l}{m^k}$ of the variable. Then by equation (4) the derivative corresponding to $\Delta'_{k,l}$ is $\alpha_l .\ m^k \Delta_{k,l}$. If $x_0$ is any arbitrary point of the interval $A$ such that

$$\frac{l_k}{m^k} \leqq x_0 < \frac{l_{k+1}}{m^k}, \quad (l_k \text{ is any integer}).$$

then not only $\Delta_{k,l_k}$ but also $\Delta'_{k,l_k}$, from its definition, correspond to an interval inclosing the point $x_0$, and we have just seen the derivatives corresponding to $\Delta_{k,l_k}$ and $\Delta'_{k,l_k}$ are $m^k \Delta_{k,l_k}$ and $\mathcal{U}_{l_k}\, m^k \Delta_{k,l_k}$. Now since $\mathcal{U}$ is always negative, these derivatives will always have opposite signs, and therefore the function will not have either a finite or infinite derivative at any point.

Finding the values of the function $f(x; 4, 4, -5, -5, 4, 4)$ as in previous examples, we have:

| | $k=0$ | | $k=1$ | | $k=2$ | | |
|---|---|---|---|---|---|---|---|
| | $x$ | $f(x)$ | $x$ | $f(x)$ | $x$ | $f(x)$ | |
| | 0 | 0 | 0 | 0 | 0 | 0 | |
| | | | | | $1/36$ | $8/3$ | |
| | | | | | $2/36$ | $16/3$ | |
| | | | | | $3/36$ | $7/3$ | |
| | | | | | $4/36$ | $-4/3$ | |
| | | | | | $5/36$ | $4/3$ | |
| | | | $1/6$ | 4 | $6/36$ | 4 | |

| h = 0 | | h = 1 | | h = 2 | | h = 0 | | h = 1 | | h = 2 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| x | f(x) | x | f(x) | x | f(x) | x | f(x) | x | f(x) | x | f(x) |
| | | | | 7/36 | 26/3 | | | | | 22/36 | 28/6 |
| | | | | 8/36 | 28/3 | | | | | 23/36 | 8/6 |
| | | | | 9/36 | 6 | | | 4/6 | -2 | 24/36 | -2 |
| | | | | 10/36 | 8/3 | | | | | 25/36 | 2/3 |
| | | | | 11/36 | 16/3 | | | | | 26/36 | 10/3 |
| | | 2/6 | 8 | 12/36 | 8 | | | | | 27/36 | 0 |
| | | | | 13/36 | 14/3 | | | | | 28/36 | -10/3 |
| | | | | 14/36 | 4/3 | | | | | 29/36 | -2/3 |
| | | | | 15/36 | 33/6 | | | 5/6 | 2 | 30/36 | 2 |
| | | | | 16/36 | 58/6 | | | | | 31/36 | 14/3 |
| | | | | 17/36 | 38/6 | | | | | 32/36 | 32/3 |
| | | 3/6 | 3 | 18/36 | 3 | | | | | 33/36 | 4 |
| | | | | 19/36 | -1/3 | | | | | 34/36 | 2/3 |
| | | | | 20/36 | -11/3 | | | | | 35/36 | 10/3 |
| | | | | 21/36 | 3/6 | 1 | 6 | 6/6 | 6 | 36/36 | 6 |

For the geometrical representation see figure 5, on the following page

Figure 5.

Unit on x axis 12

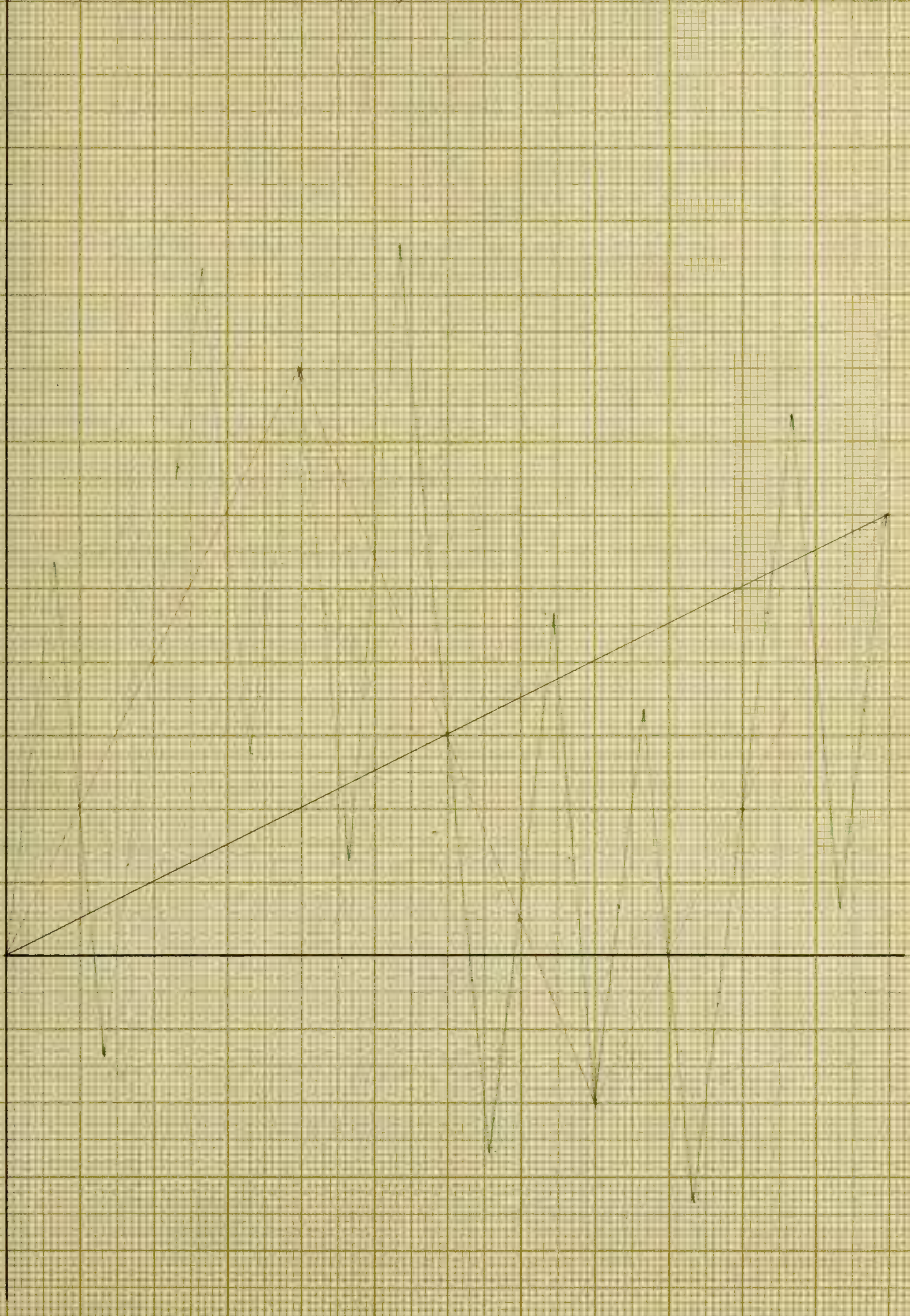
Unit on y axis 1

k = 0

k = 1

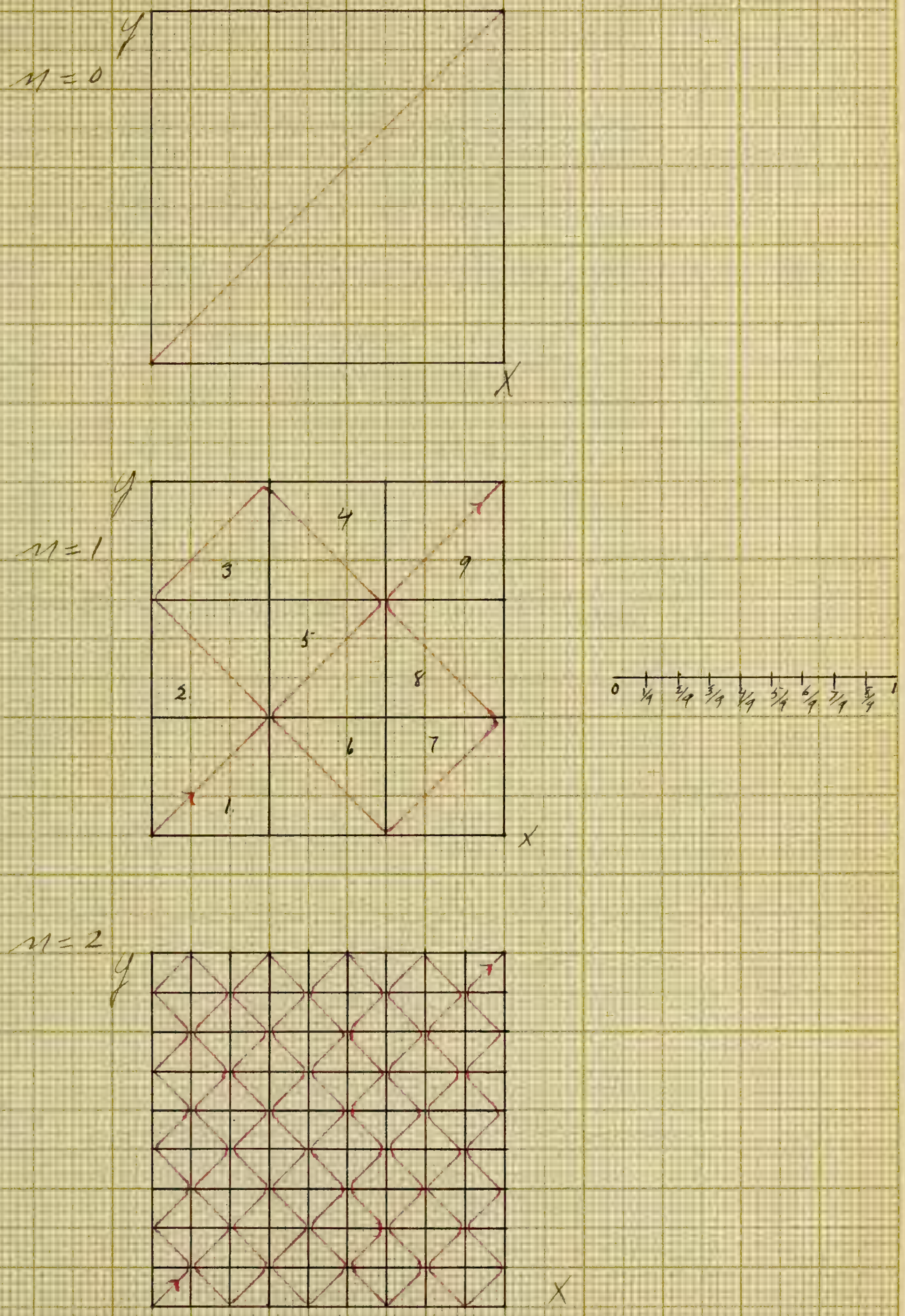
k = 2

$f(x; 4, 4, -5, -5, 4, 4)$



§ (2) Peano's method of constructing a continuous function which has at no point a derivative. Peano uses 3 as his basal number and gets a continuous curve in the following manner. Choose the segment 0 1 of a straight line and a unit square. When the line is divided into $3^{2n}$ equal squares. Let t be any point on the line within the given segment, then the squares can be assigned in a one to one manner continuously to the t points, so that as t moves continuously from 0 to 1, the corresponding point (x, y) in the square will move continuously through all the points of the square. (See figure). The locus of the point (x, y) then is: $x = \varphi(t)$, $y = \psi(t)$. The assignment of squares to points is made so that corresponding to two adjacent segments of the line, we have two adjacent squares, and the approximation curve traced by drawing the diagonals of these squares. Figure on the following page shows the first three approximations.

Figure 6.



That the curve will be defined for every value of $t$ between 0 and 1, follows directly the proof in §1 (a) page 64. The curve is continuous; for let

$$|t_1 - t_2| \leq 3^{-2n},$$

then since by our method of assignment the squares are adjacent and the points $(x, y)$, corresponding to $t_1$ and $t_2$, cannot have abscissas or ordinates of greater difference than twice the length of the side of a square. But the length of the side of a square is $\frac{1}{3^n}$ and we may therefore put

$$|\varphi(t_1) - \varphi(t_2)| \leq \frac{2}{3^n}$$

$$|\psi(t_1) - \psi(t_2)| \leq \frac{2}{3^n}.$$

Now as $n$ increases indefinitely the right hand members of these equations approach zero, and we know, from our definition of continuity on page 3, that the curve is continuous.

By finding the abscissas and ordinates of the curve in Fig. 6, for the different values of $t$, we can construct curves first with $t$ and $x$ as axes and then with $t$ and $y$ as the axes. These curves we shall call the

$tx$ and $ty$ curves. Thus for the approximation curve $n=2$ in figure 9, when $t$ has passed from 0 to $\frac{1}{9}$, $x$ has passed from 0 to $\frac{1}{3}$; when $t$ increases to $\frac{2}{9}$, $x$ decreases to 0; as $t$ increases to $\frac{3}{9}$, $x$ increases to $\frac{1}{3}$; as $t$ increases to $\frac{4}{9}$, $x$ increases to $\frac{6}{9}$; etc. Figure 7 shows the $tx$ and $ty$ curves with the $t$ axis drawn perpendicular to $xy$ plane. Figures 8, 9, 10, 11, show the separate $tx$ and $ty$ curves for $n=1$ and $n=2$.

Figure 7.

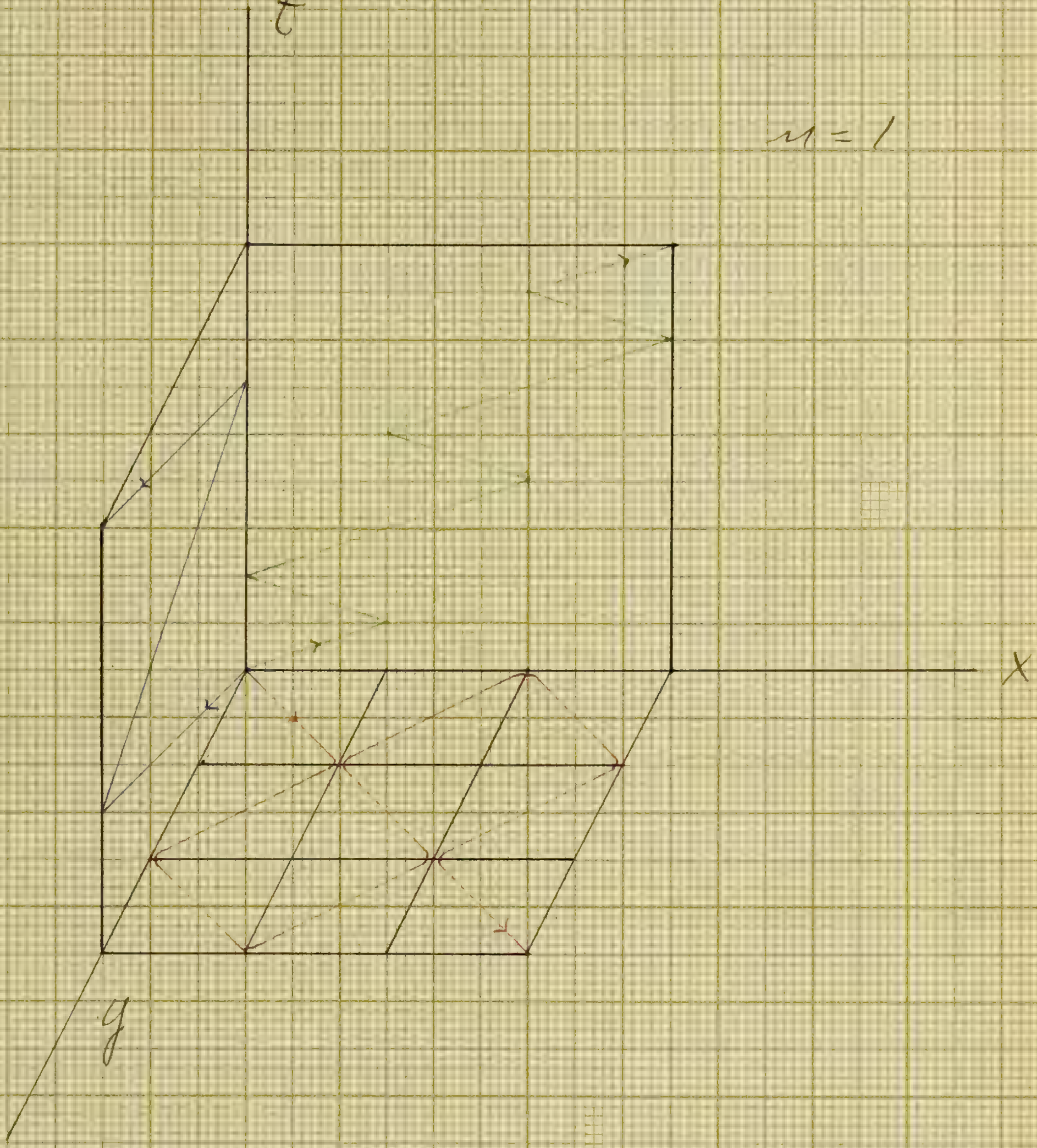


Figure 8. tx curve for $u = 1$

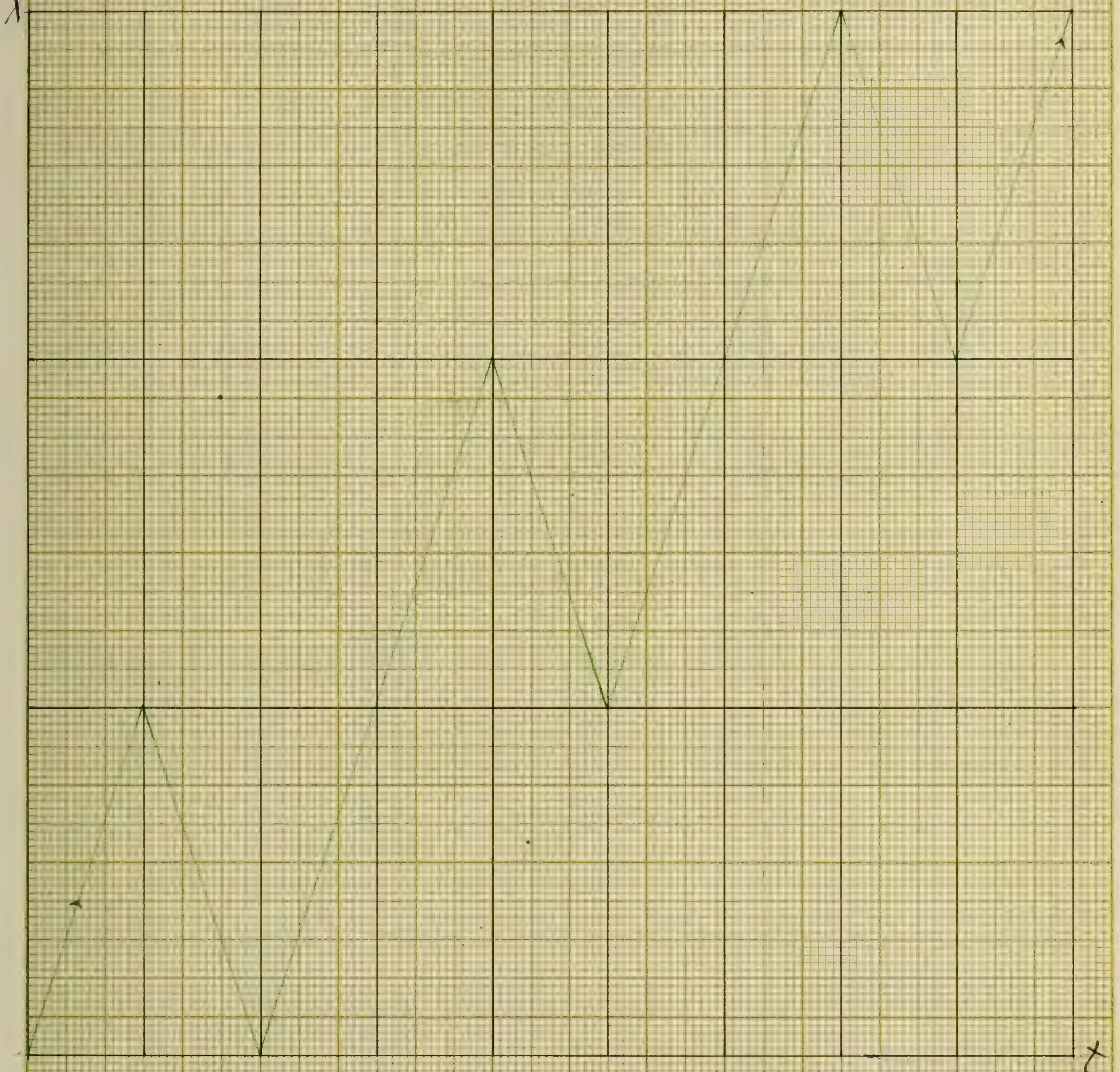


Figure 9.

$tx$ curve for $n=2$.

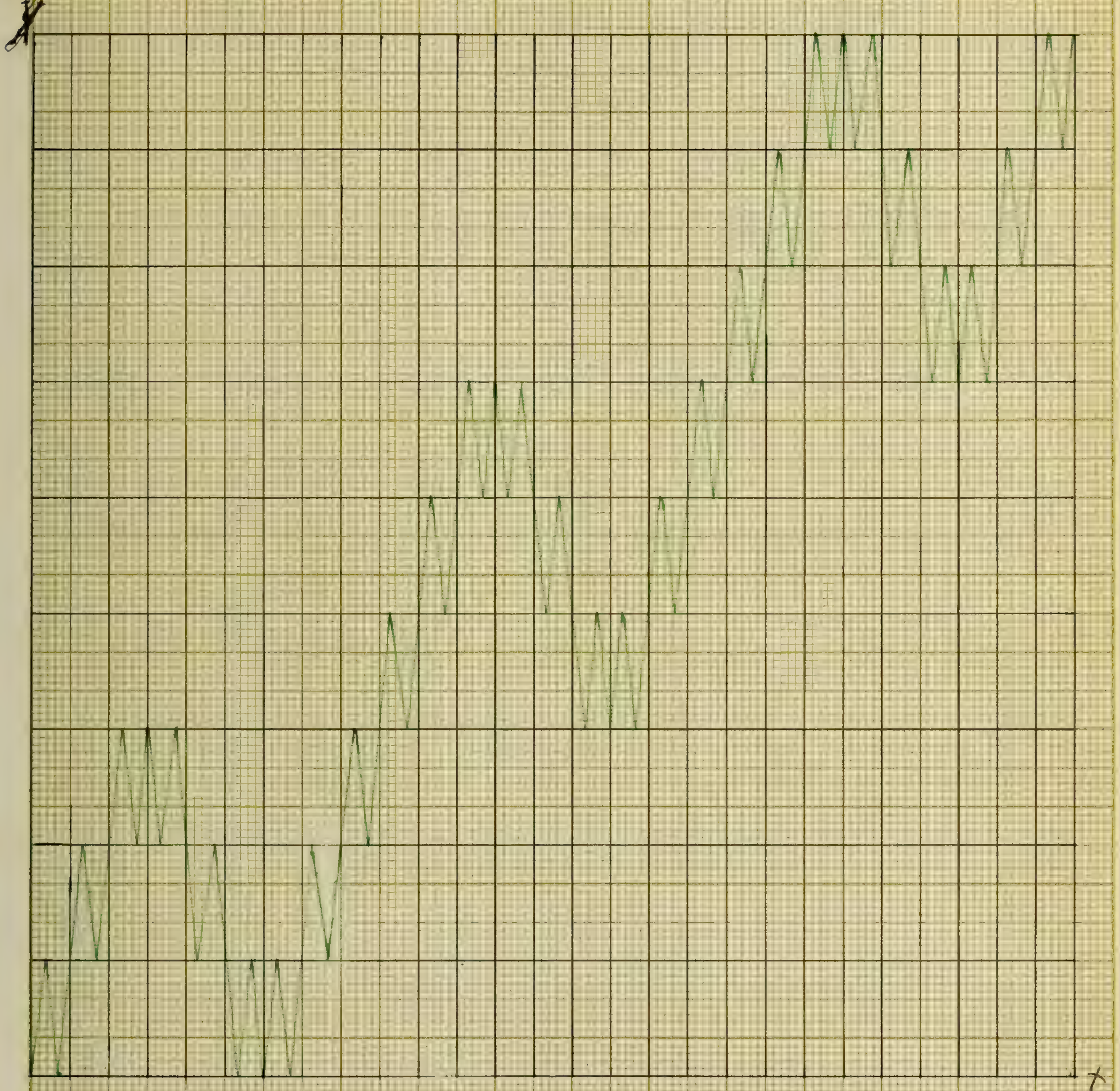


Figure 10.

t y curve for n=1

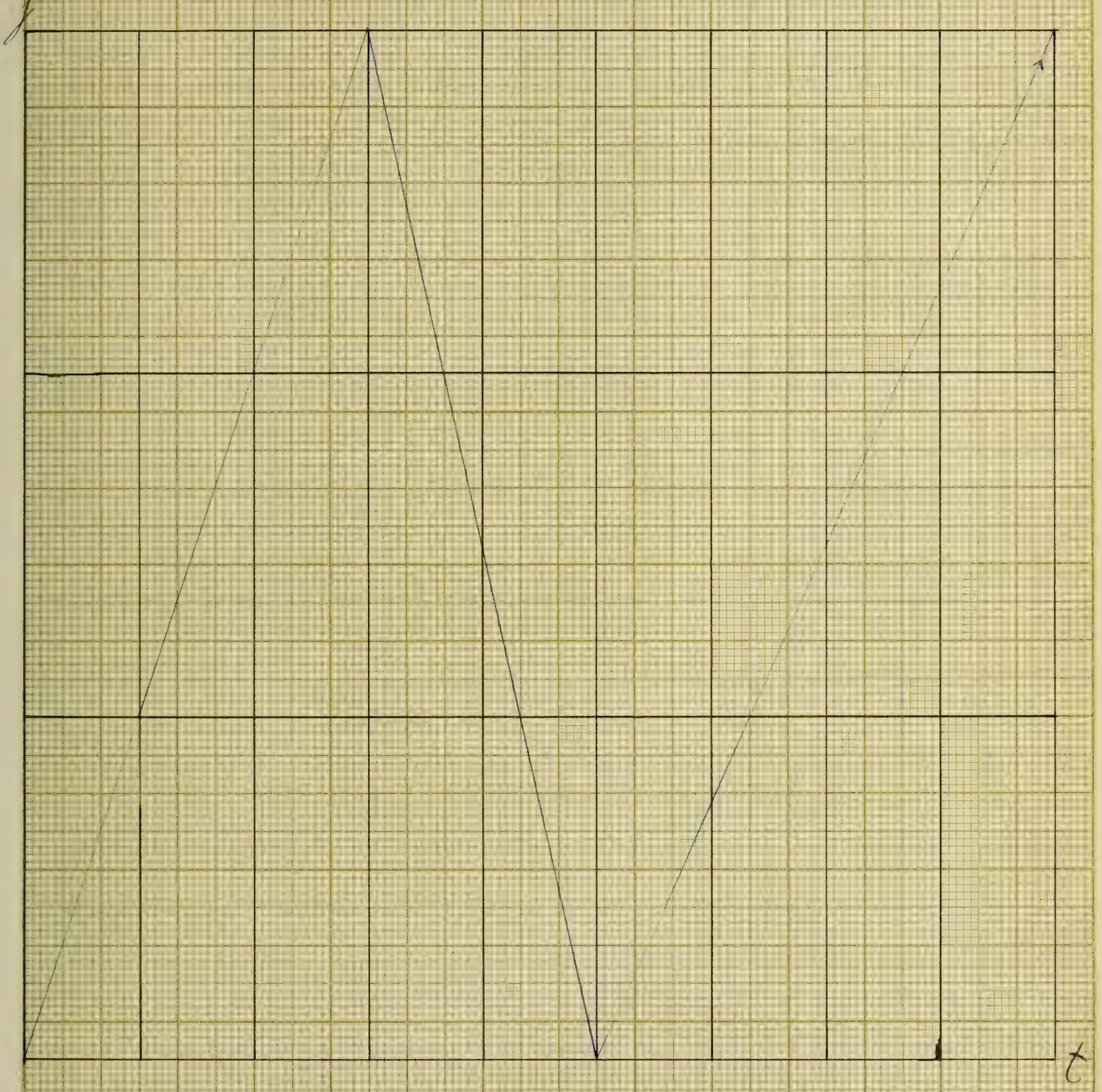
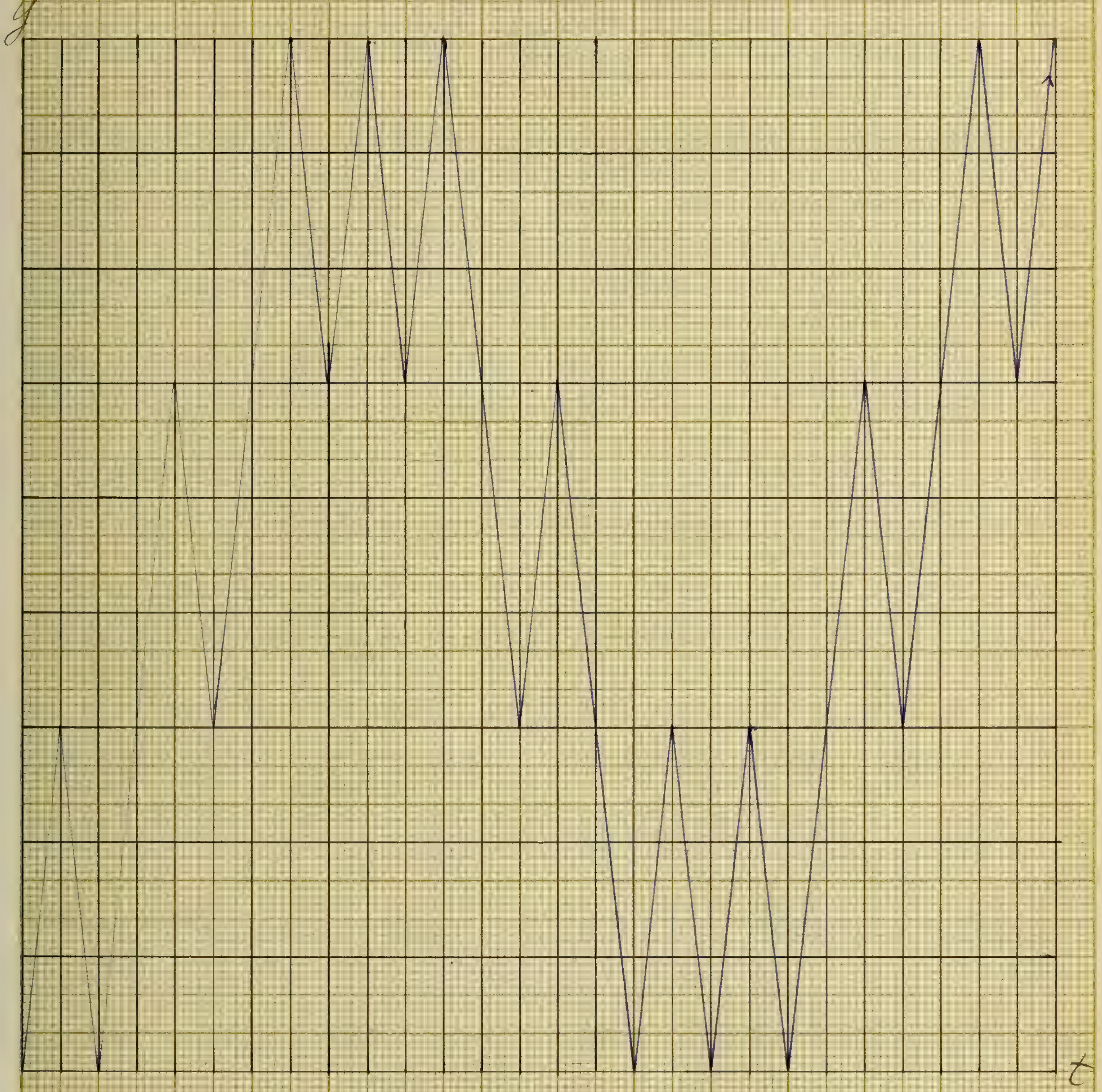


Figure 11.

t y curve for n = 2.



From figures 8 and 9, we see that for the $tx$ curve the fundamental square, $(0 \leqq t \leqq 1,\ 0 \leqq x \leqq 1)$ is divided into rectangles of the dimensions:

$$t = \frac{1}{3^{2n}}\ ,\ x = \frac{1}{3^{n}} \quad (n = 0, 1, 2 \ldots).$$

For example in figure 8, where $n=1$, the dimensions of the rectangles are

$$t = \frac{1}{9}\ ,\ x = \frac{1}{3}\ .$$

For the $ty$ curves we see from figures 10 and 11 that the fundamental square is divided into rectangles of the dimensions:

$$t = \frac{1}{3^{2n-1}}\ ,\ y = \frac{1}{3^{n-1}} \quad (n = 0, 1, 2 \ldots).$$

For example in figure 10, where $n=1$, the dimensions of the rectangles are

$$t = 1/3\ ,\ y = 1.$$

The points, where the curve changes direction, we shall call node points. The tangents to the right and to the left of these points will always be of opposite sign, and therefore we can say the tangent at the node points does not exist. These node points are everywhere dense for they exist at every rational fractional point between 0 and 1.

Then our curve does not have a tangent at any rational fractional point. Since all these rational fractional points $\frac{1}{3^n}$ are included in the interval 0 to 1, and since every point in the interval 0 to 1, can be represented as the limit of a sequence of the rational fractions, we can say from our conclusions on page 63, that the curve does not have a tangent at any point.